

FIXED POINT SUBALGEBRAS OF RATIONAL HIGHER-DIMENSIONAL NON-COMMUTATIVE TORI

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ABSTRACT. We consider the fixed point subalgebras of higher-dimensional rational non-commutative tori induced by finite order automorphisms derived from a diagonal extension of the canonical $SL(2, \mathbb{Z})$ action on rotation algebras. We give an alternative characterization and determine their K -theory.

1. INTRODUCTION

Non-commutative tori are probably the most well understood examples of non-commutative, not necessarily type I, C^* -algebras but even then several questions still remained unanswered. Firstly we would like to recall the definition of a non-commutative d -dimensional torus, \mathcal{A}_Θ [8]. Let $\Theta \in \text{Hom}(\mathbb{Z}^d \wedge \mathbb{Z}^d, \mathbb{T})$ be a character of $\mathbb{Z}^d \wedge \mathbb{Z}^d$. Then \mathcal{A}_Θ is the universal unital C^* -algebra generated by d unitary operators u_1, \dots, u_d subject to the relations $u_j u_i = \Theta(e_i \wedge e_j) u_i u_j$, $i, j = 1, \dots, d$, where $\{e_1, \dots, e_d\}$ is a standard basis for \mathbb{Z}^d . Note for $d = 2$ we shall denote \mathcal{A}_Θ by A_θ with $\theta \in [0, 1)$ since the algebra in this case is determined by a real number θ and the relation $u_2 u_1 = e^{2\pi i \theta} u_1 u_2$. This algebra is isomorphic to the crossed product C^* -algebra $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ for the action of \mathbb{Z} on $C(\mathbb{T})$ induced by the rotation by $2\pi\theta$ on the circle \mathbb{T} and consequently is also referred to as a rotation algebra. In this note we are interested in automorphisms of higher-dimensional non-commutative tori and the study of the corresponding fixed point subalgebras.

For the 2-dimensional non-commutative torus or rotation algebra, A_θ , the situation has been considered in some depth [3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17]. The automorphism group of irrational rotation algebras (up to inner automorphisms) has recently been described by Elliott and Rørdam [9] as generated by the automorphisms induced by $SL(2, \mathbb{Z})$ and an automorphism inducing a determinant -1 transformation on $K_1(A_\theta) \cong \mathbb{Z}^2$ (of which there is no explicit description).

In particular the natural action π of $SL(2, \mathbb{Z})$ on the rotation algebra A_θ , first described in [6, 17], is given by $\pi_A(u_1) = e^{\pi i a c \theta} u_1^a u_2^c$, $\pi_A(u_2) = e^{\pi i b d \theta} u_1^b u_2^d$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. Considering the fixed point algebras arising from this action there are four interesting cases [10]. These are determined by the automorphisms coming from the four non-trivial finite subgroups of $SL(2, \mathbb{Z})$, namely, up

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to conjugacy, $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ and \mathbb{Z}_6 . The generators of these cyclic groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ and \mathbb{Z}_6 can be taken to be the following elements of $SL(2, \mathbb{Z})$ respectively,

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

It would be very interesting to know the structure and classify the fixed point subalgebras and the crossed products associated to elements of $SL(2, \mathbb{Z})$.

For θ rational, $\theta = \frac{p}{q}$, there is a complete description of the fixed point subalgebras $A_\theta^{\mathbb{Z}_i}$ for all four cases $i = 2, 3, 4, 6$ [3, 10, 11, 12, 13], as trivial matrix bundle C^* -algebras over the 2-sphere S^2 with generic fiber M_q and exceptional fibers at four points of S^2 for $i = 2$ and three points of S^2 for $i = 3, 4, 6$ (at least for $q > i$; for $q \leq i$ an analogous description is true but there are fewer fixed points and their orders could be smaller). In fact a similar description is also valid for the crossed products $A_\theta \rtimes \mathbb{Z}_i$. For θ irrational, in the case of the first of these, usually referred to as the flip automorphism, a result of Bratteli and Kishimoto [4] has shown the fixed point subalgebra is in fact an AF algebra, a fact conjectured but as yet unproven for the other three cases.

These results lead naturally to ask what can be done in higher dimensions. Unfortunately the situation here is far from complete with very little known. The fact that the same is true for the automorphism group for higher dimensions is one major reason. However, some progress has been made recently. Since most non-commutative tori have just been shown to be inductive limits of circle algebras [2], the determination of the automorphism group of A_Θ may now be possible. We have already considered the generalized flip automorphism, Σ , elsewhere [14], in the completely irrational case and recently Boca has shown that the fixed point subalgebra is indeed AF [2], as in dimension two. Here we wish to consider the case of rational higher-dimensional tori where it is possible to say something more about the structure of A_Θ . Now Θ is said to be rational if $\text{Range}(\Theta)$ is a finite subgroup of \mathbb{T} and in this case there is an $n \in \mathbb{N}$ such that $\text{Range}(\Theta) = \left\{ \exp\left(\frac{2\pi ip}{n}\right) \mid p \in \mathbb{Z} \right\} \cong \mathbb{Z}_n$, which we will call the order of Θ . Throughout this note we will assume that Θ is rational.

By a result of Brenken [7], if the order of Θ is non-trivial, that is, larger than one, we can assume that \mathcal{A}_Θ is isomorphic to the following crossed product,

$$A_{1/q_1} \otimes \cdots \otimes A_{1/q_{\ell-1}} \otimes A_{r/q_\ell} \otimes C(\mathbb{T}^s)$$

where $r, q_j, j = 1, \dots, \ell$, are positive integers, $s = d - 2\ell$, and $(r, q_\ell) = 1, 1 < q_\ell | \dots | q_1, q_1 = \text{order of } \Theta$. For ease of notation in what follows we will denote by \mathcal{A}'_Θ the tensor product $A_{1/q_1} \otimes \cdots \otimes A_{1/q_{\ell-1}} \otimes A_{r/q_\ell}$. With this decomposition we can obviously generalize the action π on A_θ to an action of $SL(2, \mathbb{Z})$ on \mathcal{A}_Θ . Here we wish to consider only the diagonal situation by letting π act on each non-commutative component of the crossed product and have no action on the commutative component. That is, we will define $\Pi : SL(2, \mathbb{Z}) \rightarrow \text{Aut}(\mathcal{A}_\Theta)$ by

$$\Pi = \pi \otimes \cdots \otimes \pi \otimes id.$$

We will generalize the results for the rotation algebra to this action Π on rational non-commutative tori. This is, of course, a restriction since there are many ways to generalize π , including different actions on different components. The automorphism group, in this case, is larger than that considered. Note, however, that

in general an action of $SL(d, \mathbb{Z})$, $d > 2$, is not possible and for general higher-dimensional tori neither is the action described above unless we restrict to tensor products of rotation algebras. Both of these questions, linked with the structure of the automorphism group in general, require further study.

More precisely, building on those for the rotation algebra we will give a description of \mathcal{A}_Θ^i , $i = 2, 3, 4, 6$, as Morita equivalent to trivial matrix bundle C^* -algebras over a certain quotient space with exceptional fibers at a certain number of fixed points and hence deduce their K -theory. To save having a large list of special cases, similar to that for small values of q in the rotation algebra case, we will suppose q_j is larger than the order of the considered automorphism for all $j = 1, \dots, \ell$.

2. AN ALTERNATIVE DESCRIPTION FOR THE ALGEBRA \mathcal{A}_Θ

Firstly recall the following explicit description of the rotation algebra A_q , $(p, q) = 1$, see for example [15, 4, 11, 12, 13].

$$A_q \cong \left\{ f \in C(\mathbb{R} \times \mathbb{R}) \otimes M_q \left| \begin{array}{l} f(x, y + n) = (\alpha_{1,q})^n f(x, y), \quad 0 \leq x \leq 1, \\ f(x + m, y) = (\alpha_{2,q})^m f(x, y), \quad 0 \leq y \leq 1, \end{array} \right. \right\}$$

where $\alpha_{1,q}$ and $\alpha_{2,q}$ are the automorphisms of M_q defined by

$$\begin{aligned} \alpha_{1,q}(U_{o,q}) &= U_{o,q}, & \alpha_{1,q}(V_{o,q}) &= \omega_q V_{o,q}, \\ \alpha_{2,q}(U_{o,q}) &= \omega_q U_{o,q}, & \alpha_{2,q}(V_{o,q}) &= V_{o,q}, \end{aligned}$$

with $\omega_q = e^{\frac{2\pi i}{q}}$, $\rho_q = \omega_q^p$, $U_{o,q} = (\delta_k^t \rho_q^t)_{k,t=0,\dots,q-1}$ and $V_{o,q} = (\delta_k^{t-1(\text{mod } q)})_{k,t=0,\dots,q-1}$.

Note that $U_{o,q}$ and $V_{o,q}$ generate M_q and the two unitary generators u_1 and u_2 of A_q in this description correspond respectively to the functions,

$$U_q(x, y) = \omega_q^x U_{o,q}, \quad \text{and} \quad V_q(x, y) = \omega_q^y V_{o,q}.$$

Hence, since the elements of type $f_1(x_1, y_1) \otimes \dots \otimes f_\ell(x_\ell, y_\ell)$, $f_j \in A_{1/q_j}$, $j = 1, \dots, \ell - 1$, $f_\ell \in A_{r/q_\ell}$ generate \mathcal{A}'_Θ , we get that

$$\mathcal{A}_\Theta \cong \left\{ f \in C((\mathbb{R} \times \mathbb{R})^\ell) \otimes M_Q \otimes C(\mathbb{T}^s) \left| \begin{array}{l} f(x_1, y_1, \dots, x_j, y_j + n, \dots, x_\ell, y_\ell, t) \\ = ({}^j\tilde{\alpha}_{1,q_j})^n f(x_1, y_1, \dots, x_j, y_j, \dots, x_\ell, y_\ell, t), \\ f(x_1, y_1, \dots, x_j + m, y_j, \dots, x_\ell, y_\ell, t) \\ = ({}^j\tilde{\alpha}_{2,q_j})^m f(x_1, y_1, \dots, x_j, y_j, \dots, x_\ell, y_\ell, t), \\ 0 \leq x_j, y_j \leq 1, j = 1, \dots, \ell \end{array} \right. \right\},$$

where $Q = \prod_{j=1}^\ell q_j$ and ${}^j\tilde{\alpha}_{k,q_j} = 1 \otimes \dots \otimes \underbrace{\alpha_{k,q_j}}_{j\text{-th slot}} \otimes 1 \otimes \dots \otimes 1$ is an automorphism of

M_Q for $k = 1, 2$ and $j = 1, \dots, \ell$.

The 2ℓ unitary generators of \mathcal{A}'_Θ in the above description correspond to the functions

$$\begin{aligned} U_{q_j} &= 1 \otimes \dots \otimes 1 \otimes \underbrace{\omega_{q_j}^{x_j} U_{o,q_j}}_{j\text{-th slot}} \otimes 1 \otimes \dots \otimes 1, \quad j = 1, \dots, \ell, \\ V_{q_j} &= 1 \otimes \dots \otimes 1 \otimes \underbrace{\omega_{q_j}^{y_j} V_{o,q_j}}_{j\text{-th slot}} \otimes 1 \otimes \dots \otimes 1, \quad j = 1, \dots, \ell. \end{aligned}$$

3. THE FIXED POINT SUBALGEBRA \mathcal{A}_Θ^S

The generator $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of the subgroup \mathbb{Z}_2 of $SL(2, \mathbb{Z})$ corresponds to the automorphism S of \mathcal{A}_Θ given by

$$(Sf)(x_1, y_1, \dots, x_\ell, y_\ell, t) = S_0(f(1 - x_1, 1 - y_1, \dots, 1 - x_\ell, 1 - y_\ell, t)),$$

where $S_0 = \sigma_{o, q_1} \otimes \dots \otimes \sigma_{o, q_\ell}$ and $\sigma_{o, q}$, $q \in \mathbb{N}$, is the automorphism of M_q determined by,

$$\sigma_{o, q}(U_{o, q}) = \omega_q^* U_{o, q}^*, \quad \sigma_{o, q}(V_{o, q}) = \omega_q^* V_{o, q}^*.$$

(It is in fact an easy computation to check this.) For $d = 2$, $\ell = 1$, $s = 0$ the above expression for S was introduced in [4] and in this more general case the analysis will follow a similar procedure to that paper. It is also clear that S also induces an automorphism, which we will still call S , of \mathcal{A}'_Θ , since it acts as the identity on $C(\mathbb{T}^s)$. Moreover the fixed point algebra \mathcal{A}_Θ^S is isomorphic to $\mathcal{A}'_\Theta^S \otimes C(\mathbb{T}^s)$. We will therefore proceed to describe \mathcal{A}'_Θ^S .

To describe the fixed point algebra \mathcal{A}'_Θ^S the idea is to make use of the splitting of S into an automorphism of M_Q and a homeomorphism of the underlying space introduced above to characterize \mathcal{A}'_Θ^S as a C^* -matrix bundle over the quotient space, which is an orbifold. More precisely, let σ be the homeomorphism of $(\mathbb{R} \times \mathbb{R})^\ell$ given by

$$(x_1, y_1, \dots, x_\ell, y_\ell) \xrightarrow{\sigma} (1 - x_1, 1 - y_1, \dots, 1 - x_\ell, 1 - y_\ell).$$

Note that σ also induces an order two homeomorphism of $\mathbb{T}^{2\ell}$, which we will still call σ . Moreover,

$$\mathbb{T}^{2\ell}/\sigma = \left\{ \begin{array}{l} (x_1, y_1, \dots, x_\ell, y_\ell) \\ \in ([0, 1] \times [0, 1])^\ell \end{array} \middle| y_1 \leq x_1 \text{ with the equivalences below} \right\},$$

for $x_1 = 1$: $(1, y_1, x_2, y_2, \dots, x_\ell, y_\ell) \sim (1, 1 - y_1, \dots, 1 - x_\ell, 1 - y_\ell)$, for $y_1 = 0$: $(x_1, 0, x_2, y_2, \dots, x_\ell, y_\ell) \sim (1 - x_1, 0, \dots, 1 - x_\ell, 1 - y_\ell)$, for $y_1 = x_1$: $(x_1, x_1, x_2, y_2, \dots, x_\ell, y_\ell) \sim (1 - x_1, 1 - x_1, \dots, 1 - x_\ell, 1 - y_\ell)$, and for $i = 1, \dots, \ell$:

$$\begin{aligned} (x_1, y_1, \dots, 1, y_i, \dots, x_\ell, y_\ell) &\sim (x_1, y_1, \dots, 0, y_i, \dots, x_\ell, y_\ell), \\ (x_1, y_1, \dots, x_i, 1, \dots, x_\ell, y_\ell) &\sim (x_1, y_1, \dots, x_i, 0, \dots, x_\ell, y_\ell). \end{aligned}$$

Hence the fixed point algebra \mathcal{A}'_Θ^S is isomorphic to

$$\mathcal{A}'_\Theta^S \cong \{f \in C(([0, 1] \times [0, 1])^\ell \otimes M_Q \text{ with } y_1 \leq x_1) \text{ with the conditions below}\}.$$

For $x_1 = 1$: $f(1, y_1, x_2, y_2, \dots, x_\ell, y_\ell) = {}^1\tilde{\alpha}_{2, q_1}^{-1} S_0 f(1, 1 - y_1, \dots, 1 - x_\ell, 1 - y_\ell)$, for $y_1 = 0$: $f(x_1, 0, x_2, y_2, \dots, x_\ell, y_\ell) = {}^1\tilde{\alpha}_{1, q_1} S_0 f(1 - x_1, 0, \dots, 1 - x_\ell, 1 - y_\ell)$, for $y_1 = x_1$: $f(x_1, x_1, x_2, y_2, \dots, x_\ell, y_\ell) = S_0 f(1 - x_1, 1 - x_1, \dots, 1 - x_\ell, 1 - y_\ell)$, and for $i = 1, \dots, \ell$:

$$\begin{aligned} f(x_1, y_1, \dots, 1, y_i, \dots, x_\ell, y_\ell) &= {}^i\tilde{\alpha}_{2, q_i} f(x_1, y_1, \dots, 0, y_i, \dots, x_\ell, y_\ell), \\ f(x_1, y_1, \dots, x_i, 1, \dots, x_\ell, y_\ell) &= {}^i\tilde{\alpha}_{1, q_i} f(x_1, y_1, \dots, x_i, 0, \dots, x_\ell, y_\ell). \end{aligned}$$

With analogous computations to the case of rotation algebras (cf. [4, 11, 12, 13]) the fixed point subalgebra is shown to be Morita equivalent to

$\mathcal{A}_\Theta^S \sim \{f \in C(\mathbb{T}^{2\ell}/\sigma) \otimes M_Q \mid f(\alpha_j) \text{ is 2-block diagonal, } j = 1, \dots, 2^{2\ell}\} \otimes C(\mathbb{T}^s)$, where $\alpha_j, j = 1, \dots, 2^{2\ell}$, are the (order 2) elliptic points of $\mathbb{T}^{2\ell}/\sigma$, that is, the images of the (order 2) fixed points of the automorphism σ on $\mathbb{T}^{2\ell}$. Namely, $\alpha_j = [A_{j_1}, \dots, A_{j_\ell}]$ with $A_{j_k} \in \{(0, 0), (1/2, 1/2), (1/2, 0), (1, 1/2)\}$ ($[\]$ denotes the equivalence class of the given point in the quotient orbifold). The fact that these are the only elliptic points of $\mathbb{T}^{2\ell}/\sigma$ follows from analogous facts as those for dimension two.

4. THE FIXED POINT SUBALGEBRA \mathcal{A}_Θ^T

The generator $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of the subgroup \mathbb{Z}_4 of $SL(2, \mathbb{Z})$ corresponds to the automorphism T of \mathcal{A}_Θ given by

$$(Tf)(x_1, y_1, \dots, x_\ell, y_\ell, t) = T_0(f(1 - y_1, x_1, \dots, 1 - y_\ell, x_\ell, t)),$$

where $T_0 = \tau_{o, q_1} \otimes \dots \otimes \tau_{o, q_\ell}$ and $\tau_{o, q}$ is the automorphism of M_q determined by

$$\tau_{o, q}(U_{o, q}) = \omega_q^* V_{o, q}^*, \quad \tau_{o, q}(V_{o, q}) = U_{o, q}.$$

As in the previous case we use this splitting of T to characterize \mathcal{A}_Θ^T as a C^* -matrix bundle over the quotient space, which is an orbifold. More precisely, let τ be the homeomorphism of $(\mathbb{R} \times \mathbb{R})^\ell$ given by

$$(x_1, y_1, \dots, x_\ell, y_\ell) \xrightarrow{\tau} (1 - y_1, x_1, 1 - y_2, x_2, \dots, 1 - y_\ell, x_\ell).$$

This also induces a homeomorphism of $\mathbb{T}^{2\ell}$, which we will still call τ and note that $\tau^2 = \sigma$. Now,

$$\mathbb{T}^{2\ell}/\tau = \left\{ (x_1, y_1, \dots, x_\ell, y_\ell) \mid \begin{array}{l} y_1 \leq \min\{x_1, 1 - x_1\} \text{ with the equivalences below} \\ \in ([0, 1] \times [0, 1])^\ell \end{array} \right\},$$

for $y_1 = 0$: $(x_1, 0, x_2, y_2, \dots, x_\ell, y_\ell) \sim (1 - x_1, 0, 1 - x_2, 1 - y_2, \dots, 1 - x_\ell, 1 - y_\ell)$,
 for $y_1 = x_1 \leq 1/2$: $(x_1, x_1, x_2, y_2, \dots, x_\ell, y_\ell) \sim (1 - x_1, x_1, 1 - y_2, x_2, \dots, 1 - y_\ell, x_\ell)$,
 and for $i = 1, \dots, \ell$:

$$\begin{aligned} (x_1, y_1, \dots, 1, y_i, \dots, x_\ell, y_\ell) &\sim (x_1, y_1, \dots, 0, y_i, \dots, x_\ell, y_\ell), \\ (x_1, y_1, \dots, x_i, 1, \dots, x_\ell, y_\ell) &\sim (x_1, y_1, \dots, x_i, 0, \dots, x_\ell, y_\ell). \end{aligned}$$

Hence the fixed point algebra \mathcal{A}_Θ^T is isomorphic to

$$\mathcal{A}_\Theta^T \cong \{f \in C(([0, 1] \times [0, 1])^\ell \otimes M_Q, y_1 \leq \min\{x_1, 1 - x_1\}) : \text{the conditions below}\}.$$

For $y_1 = 0$: $f(x_1, 0, x_2, y_2, \dots, x_\ell, y_\ell) = {}^1\tilde{\alpha}_{1, q} S_0 f(1 - x_1, 0, \dots, 1 - x_\ell, 1 - y_\ell)$, for
 $y_1 = x_1 \leq 1/2$: $f(x_1, x_1, x_2, y_2, \dots, x_\ell, y_\ell) = T_0 f(1 - x_1, x_1, \dots, 1 - y_\ell, x_\ell)$, and for
 $i = 1, \dots, \ell$:

$$\begin{aligned} f(x_1, y_1, \dots, 1, y_i, \dots, x_\ell, y_\ell) &= {}^i\tilde{\alpha}_{2, q_i} f(x_1, y_1, \dots, 0, y_i, \dots, x_\ell, y_\ell), \\ f(x_1, y_1, \dots, x_i, 1, \dots, x_\ell, y_\ell) &= {}^i\tilde{\alpha}_{1, q_i} f(x_1, y_1, \dots, x_i, 0, \dots, x_\ell, y_\ell), \end{aligned}$$

The elliptic points of $\mathbb{T}^{2\ell}/\tau$ are of the following types:

Order 4. $\alpha_j = [A_{j_1}, \dots, A_{j_\ell}]$ where $A_{j_k} \in \{(1/2, 1/2), (0, 0)\}$. There are 2^ℓ distinct such points in \mathbb{T}^{2^ℓ}/τ .

Order 2. $\beta_j = [B_{j_1}, \dots, B_{j_\ell}]$ where $B_{j_k} \in \{(1/2, 1/2), (0, 0), (1, 1/2), (1/2, 0)\}$ with at least one of the B_{j_k} in each β_j different from $(1/2, 1/2)$ and $(0, 0)$ (i.e. each β_j is not an α_j). There are $(2^{2^\ell} - 2^\ell)/2$ distinct such points in the quotient orbifold \mathbb{T}^{2^ℓ}/τ .

With analogous computations to the case of rotation algebras the fixed point subalgebra is shown to be Morita equivalent to

$$\mathcal{A}_\Theta^T \sim \left\{ \begin{array}{l} f \in C(\mathbb{T}^{2^\ell}/\tau) \mid f(\alpha_i) \text{ is 4-block diagonal, } i=1, \dots, 2^\ell, \\ \otimes M_Q \quad \mid f(\beta_j) \text{ is 2-block diagonal, } j=1, \dots, (2^{2^\ell} - 2^\ell)/2 \end{array} \right\} \otimes C(\mathbb{T}^s),$$

where α_i , $i = 1, \dots, 2^\ell$, are the order 4 elliptic points of \mathbb{T}^{2^ℓ}/τ and β_j , $j = 1, \dots, (2^{2^\ell} - 2^\ell)/2$, are the order two elliptic points.

5. THE FIXED POINT SUBALGEBRA \mathcal{A}_Θ^Z

The generator $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ of the subgroup \mathbb{Z}_3 of $SL(2, \mathbb{Z})$ corresponds to the automorphism Z of \mathcal{A}_Θ given by

$$(Zf)(x_1, y_1, \dots, x_\ell, y_\ell, t) = Z_0(f(\delta_1 - x_1 + y_1, -x_1, \dots, \delta_\ell - x_\ell + y_\ell, -x_\ell, t)),$$

where $Z_0 = \zeta_{o, q_1} \otimes \dots \otimes \zeta_{o, q_\ell}$ and ζ_{o, q_j} is the automorphism of M_{q_j} determined by

$$\zeta_{o, q_j}(U_{o, q_j}) = e^{\pi i [1 - \delta_j] \epsilon_j / (2q_j)} U_{o, q_j}^* V_{o, q_j}, \quad \zeta_{o, q_j}(V_{o, q_j}) = U_{o, q_j}^*,$$

where $\delta_j = [-1 + (-1)^{q_j}] \epsilon_j / 4$, with $\epsilon_j = 1$ for $j = 1, \dots, \ell - 1$ and $\epsilon_\ell = r$. Let ζ denote the homeomorphism of \mathbb{T}^{2^ℓ} induced by the transformation of $(\mathbb{R} \times \mathbb{R})^\ell$ given by

$$(x_1, y_1, \dots, x_\ell, y_\ell) \xrightarrow{\zeta} (\delta_1 - x_1 + y_1, -x_1, \dots, \delta_\ell - x_\ell + y_\ell, -x_\ell).$$

A domain for ζ , similar to σ and τ , can be easily computed, but its description is rather lengthy so it will not be given. The (order 3) elliptic points of the orbifold $\mathbb{T}^{2^\ell}/\zeta$ are given by $\alpha_j = [A_{j_1}, \dots, A_{j_\ell}]$ where for $k = 1, \dots, \ell - 1$, $A_{j_k} \in \{(0, 0), (2/3, 1/3), (1/3, 2/3)\}$ if q_k even or $A_{j_k} \in \{(-1/6, 1/6), (1/2, 1/2), (1/6, 5/6)\}$ if q_k odd, while $A_{j_\ell} \in \{(0, 0), (2/3, 1/3), (1/3, 2/3)\}$ if either q_ℓ or r is even or $A_{j_\ell} \in \{(-1/6, 1/6), (1/2, 1/2), (1/6, 5/6)\}$ if q_ℓ and r are both odd. There are 3^ℓ elliptic points in $\mathbb{T}^{2^\ell}/\zeta$.

Hence \mathcal{A}_Θ^Z is Morita equivalent to

$$\mathcal{A}_\Theta^Z \sim \{f \in C(\mathbb{T}^{2^\ell}/\zeta) \otimes M_Q \mid f(\alpha_i) \text{ is 3-block diagonal, } i = 1, \dots, 3^\ell\} \otimes C(\mathbb{T}^s)$$

where α_i , $i = 1, \dots, 3^\ell$, are the order 3 elliptic points of $\mathbb{T}^{2^\ell}/\zeta$.

6. THE FIXED POINT SUBALGEBRA \mathcal{A}_Θ^E

The generator $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ of the subgroup \mathbb{Z}_6 of $SL(2, \mathbb{Z})$ corresponds to the automorphism E of \mathcal{A}_Θ given by

$$(Ef)(x_1, y_1, \dots, x_\ell, y_\ell, t) = E_0(f(y_1, y_1 - x_1 + \delta_1, \dots, y_\ell, y_\ell - x_\ell + \delta_\ell, t)),$$

where $E_0 = \eta_{o,q_1} \otimes \cdots \otimes \eta_{o,q_\ell}$ and η_{o,q_j} is the automorphism of M_{q_j} determined by

$$\eta_{o,q_j}(U_{o,q_j}) = V_{o,q_j}, \quad \eta_{o,q_j}(V_{o,q_j}) = e^{\pi i[1-\delta_j]\epsilon_j/(2q_j)} U_{o,q_j}^* V_{o,q_j}.$$

Let η denote the homeomorphism of $\mathbb{T}^{2\ell}$ induced by the transformation of $(\mathbb{R} \times \mathbb{R})^\ell$ given by

$$(x_1, y_1, \dots, x_\ell, y_\ell) \xrightarrow{\eta} (y_1, y_1 - x_1 + \delta_1, \dots, y_\ell, y_\ell - x_\ell + \delta_\ell).$$

The elliptic points of $\mathbb{T}^{2\ell}/\eta$ are as follows:

Order 6. $\alpha = [A_1, \dots, A_\ell]$ where for $k = 1, \dots, \ell - 1$, $A_k = (0, 0)$ if q_k even or $A_k = (1/2, 1/2)$ if q_k odd, while $A_\ell = (0, 0)$ if either q_ℓ or r is even and $A_\ell = (1/2, 1/2)$ if q_ℓ and r are both odd.

Order 3. $\beta_j = [B_{j_1}, \dots, B_{j_\ell}]$ where for $k = 1, \dots, \ell - 1$, $B_{j_k} \in \{(0, 0), (1/3, 2/3), (2/3, 1/3)\}$ if q_k even, or $B_{j_k} \in \{(1/2, 1/2), (1/6, 5/6), (-1/6, 1/6)\}$ if q_k odd, while $B_{j_\ell} \in \{(0, 0), (1/3, 2/3), (2/3, 1/3)\}$ if either q_ℓ or r is even and $B_{j_\ell} \in \{(1/2, 1/2), (1/6, 5/6), (-1/6, 1/6)\}$ if q_ℓ and r are both odd. Also at least one of the B_{j_k} in each β_j is different from $(1/2, 1/2)$ and $(0, 0)$ (i.e. each β_j is not α).

Order 2. $\gamma_j = [C_{j_1}, \dots, C_{j_\ell}]$ where $C_{j_k} \in \{(1/2, 1/2), (0, 0), (1, 1/2), (1/2, 0)\}$ with at least one of the C_{j_k} in each γ_j different from $(1/2, 1/2)$ and $(0, 0)$ (i.e. each γ_j is not α).

In the quotient orbifold $\mathbb{T}^{2\ell}/\eta$ there is just one order 6 elliptic point, while there are $(3^\ell - 1)/2$ order 3 elliptic points and $(4^\ell - 1)/3$ order 2 ones. Hence \mathcal{A}_Θ^E is Morita equivalent to

$$\mathcal{A}_\Theta^E \sim \left\{ \begin{array}{l} f \in C(\mathbb{T}^{2\ell}/\eta) \\ \otimes M_Q \end{array} \left| \begin{array}{l} f(\alpha) \text{ is 6-block diagonal,} \\ f(\beta_i) \text{ is 3-block diagonal, } i = 1, \dots, (3^\ell - 1)/2, \\ f(\gamma_j) \text{ is 2-block diagonal, } j = 1, \dots, (4^\ell - 1)/3 \end{array} \right. \right\} \otimes C(\mathbb{T}^s),$$

where α is the order 6 elliptic point, $\beta_i, i = 1, \dots, (3^\ell - 1)/2$, are the order 3 elliptic points, and $\gamma_j, j = 1, \dots, (4^\ell - 1)/3$, are the order 2 elliptic points.

7. K-THEORY

In this section we would like to compute the K -theory for these fixed point subalgebras, that is,

$$K_j(\mathcal{A}_\Theta^X), \quad j = 0, 1, \quad X = S, T, Z, E.$$

Since all of these algebras are described in terms of the tensor product of a matrix bundle C^* -algebra with $C(\mathbb{T}^s)$ the computation reduces to the computation of the K -theory of this matrix bundle C^* -algebra and an application of K nneth's theorem. To facilitate the description of the necessary computations we will make the following definitions. Let

$$\mathcal{A}_0^S = \{f \in C(\mathbb{T}^{2\ell}/\sigma) \otimes M_Q \mid f(\alpha_i) \text{ is 2-block diagonal, } i = 1, \dots, 2^{2\ell}\},$$

$$\mathcal{A}_0^T = \left\{ f \in C(\mathbb{T}^{2\ell}/\tau) \otimes M_Q \left| \begin{array}{l} f(\alpha_i) \text{ is 4-block diagonal, } i = 1, \dots, 2^\ell, \\ f(\beta_j) \text{ is 2-block diagonal, } j = 1, \dots, (2^{2\ell} - 2^\ell)/2 \end{array} \right. \right\},$$

$$\mathcal{A}_0^Z = \{f \in C(\mathbb{T}^{2\ell}/\zeta) \otimes M_Q \mid f(\alpha_i) \text{ is 3-block diagonal, } i = 1, \dots, 3^\ell\}, \text{ and}$$

$$\mathcal{A}_0^E = \left\{ f \in C(\mathbb{T}^{2^\ell}/\eta) \otimes M_Q \left| \begin{array}{l} f(\alpha) \text{ is 6-block diagonal,} \\ f(\beta_i) \text{ is 3-block diagonal, } i = 1, \dots, (3^\ell - 1)/2, \\ f(\gamma_j) \text{ is 2-block diagonal, } j = 1, \dots, (4^\ell - 1)/3 \end{array} \right. \right\}.$$

Note that \mathcal{A}_0^X is Morita equivalent to the C^* -algebra $\mathcal{A}'_{\mathcal{O}}^X$, $X = S, T, Z$ and E which we introduced above. We will consider the computation in detail for \mathcal{A}_0^S only, the other three cases requiring analogous calculations. In \mathcal{A}_0^S we can assume that all the singular points α_i , $i = 1, \dots, 2^{2^\ell}$, are inside a small open ball D . The C^* -algebra $\mathcal{D} = \{f \in C_0(D) \otimes M_Q \mid f(\alpha_i) \text{ is 2-block diagonal, } i = 1, \dots, 2^{2^\ell}\}$ is an ideal of \mathcal{A}_0^S and the standard six-term exact sequence of K-theory associated to it is given below.

$$\begin{array}{ccccc} K_0(\mathcal{D}) & \longrightarrow & K_0(\mathcal{A}_0^S) & \longrightarrow & K_0(C(\mathbb{T}^{2^\ell}/\sigma - D) \otimes M_Q) \\ \partial \uparrow & & & & \partial \downarrow \\ K_1(C(\mathbb{T}^{2^\ell}/\sigma - D) \otimes M_Q) & \longleftarrow & K_1(\mathcal{A}_0^S) & \longleftarrow & K_1(\mathcal{D}) \end{array}$$

It is easily seen that $K_0(\mathcal{D}) \cong \mathbb{Z}^{2^{2^\ell}+1}$ and $K_1(\mathcal{D}) = 0$, so the sequence may be written as follows.

$$\begin{array}{ccccc} \mathbb{Z}^{2^{2^\ell}+1} & \longrightarrow & K_0(\mathcal{A}_0^S) & \longrightarrow & K_0(C(\mathbb{T}^{2^\ell}/\sigma - D) \otimes M_Q) \\ \partial \uparrow & & & & \partial \downarrow \\ K_1(C(\mathbb{T}^{2^\ell}/\sigma - D) \otimes M_Q) & \longleftarrow & K_1(\mathcal{A}_0^S) & \longleftarrow & 0 \end{array}$$

Now note that $\mathbb{T}^{2^\ell}/\sigma$ is an orbifold having as singular set the elliptic points α_i , $i = 1, \dots, 2^{2^\ell}$. (This can be seen by solving for the fixed points of σ .) A neighborhood of α_i can be described as $\mathbb{R}^{2^\ell}/\mathbb{Z}_2$ and hence it can be given a manifold structure since it is homeomorphic to \mathbb{R}^{2^ℓ} itself. It follows that $\mathbb{T}^{2^\ell}/\sigma$ can also be given a manifold structure. Next note that the map $\mathbb{Z}^{2^{2^\ell}+1} \rightarrow K_0(\mathcal{A}_0^S)$ is injective since the

K -theory class corresponding to the fundamental cocycle maps to $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$ in $K_0(\mathcal{A}_0^S)$ and each of the other generators correspond to block-diagonal projections of type $\begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$ at α_i whose image still give rise to generators in $K_0(\mathcal{A}_0^S)$.

Hence it follows that

$$K_0(\mathcal{A}_0^S) \cong \mathbb{Z}^{2^{2^\ell}} \oplus K^0(\mathbb{T}^{2^\ell}/\sigma), \quad K_1(\mathcal{A}_0^S) \cong K^1(\mathbb{T}^{2^\ell}/\sigma).$$

Analogously, for the other three cases, one can prove,

$$\begin{aligned} K_0(\mathcal{A}_0^T) &\cong \mathbb{Z}^{(2^{2^\ell}-2^\ell)/2} \oplus \mathbb{Z}^{3 \cdot 2^\ell} \oplus K^0(\mathbb{T}^{2^\ell}/\tau), \\ K_0(\mathcal{A}_0^Z) &\cong \mathbb{Z}^{2 \cdot 3^\ell} \oplus K^0(\mathbb{T}^{2^\ell}/\zeta), \\ K_0(\mathcal{A}_0^E) &\cong \mathbb{Z}^{(4^\ell-1)/3} \oplus \mathbb{Z}^{2 \cdot (3^\ell-1)/2} \oplus \mathbb{Z}^5 \oplus K^0(\mathbb{T}^{2^\ell}/\eta), \\ K_1(\mathcal{A}_0^X) &\cong K^1(\mathbb{T}^{2^\ell}/\alpha), \text{ where } X = T, Z, E \text{ and } \alpha = \tau, \zeta, \eta \text{ respectively.} \end{aligned}$$

Note also that $K^0(\mathbb{T}^{2\ell}/\sigma) = H^{ev}(\mathbb{T}^{2\ell})^\sigma \cong \mathbb{Z}^{2^{2\ell-1}}$ and $K^1(\mathbb{T}^{2\ell}/\sigma) = H^{od}(\mathbb{T}^{2\ell})^\sigma = 0$. Moreover $K^0(\mathbb{T}^{2\ell}/\alpha) = H^{ev}(\mathbb{T}^{2\ell})^\alpha \cong \mathbb{Z}^{2^\ell}$ and $K^1(\mathbb{T}^{2\ell}/\alpha) = H^{od}(\mathbb{T}^{2\ell})^\alpha = 0$, where $\alpha = \tau, \zeta$ and η . Hence,

$$\begin{aligned} K_0(\mathcal{A}_\Theta^S) &\cong K_1(\mathcal{A}_\Theta^S) \cong \mathbb{Z}^{3 \cdot 2^{2\ell-1}} \otimes \mathbb{Z}^{2^{s-1}}, \\ K_0(\mathcal{A}_\Theta^T) &\cong K_1(\mathcal{A}_\Theta^T) \cong \mathbb{Z}^{2^{2\ell-1} + 7 \cdot 2^{\ell-1}} \otimes \mathbb{Z}^{2^{s-1}}, \\ K_0(\mathcal{A}_\Theta^Z) &\cong K_1(\mathcal{A}_\Theta^Z) \cong \mathbb{Z}^{2 \cdot 3^\ell + 2^\ell} \otimes \mathbb{Z}^{2^{s-1}}, \\ K_0(\mathcal{A}_\Theta^E) &\cong K_1(\mathcal{A}_\Theta^E) \cong \mathbb{Z}^{3^\ell + 2^\ell + 4 + ((4^\ell - 1)/3)} \otimes \mathbb{Z}^{2^{s-1}}. \end{aligned}$$

Note that the results of the above calculations for $K_0(\mathcal{A}_\Theta^S)$ and $K_1(\mathcal{A}_\Theta^S)$ are the same as for Θ completely irrational [14].

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