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A BIALGEBRA THAT ADMITS A HOPF-GALOIS EXTENSION IS A HOPF ALGEBRA

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ABSTRACT. Let k be a commutative ring. Assume that H is a k-bialgebra, and A is an H-Galois extension of its coinvariant subalgebra B. Provided A is faithfully flat over k, we show that H is necessarily a Hopf algebra.

Throughout the paper, let k denote a fixed commutative ring. $\otimes$ means $\otimes_k$, $\mathrm{Hom}(V, W)$ means $\mathrm{Hom}_k(V, W)$, and all maps are k-linear. We use [3] as a general reference for Hopf algebra theory. Our version of Sweedler's notation is $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for comultiplications, $\delta(a) = a_{(0)} \otimes a_{(1)}$ for right comodule structures.

Let H be a k-bialgebra. A right H-comodule algebra A is said to be an H-Galois extension of $B := A^{\mathrm{co}\,H} := \{a \in A | a_{(0)} \otimes a_{(1)} = a \otimes 1\}$, if the Galois map $\kappa : A \otimes_B A \to A \otimes H$ defined by $\kappa(x \otimes y) = xy_{(0)} \otimes y_{(1)}$ is a bijection. The notion of a Hopf-Galois extension provides a unifying framework for the study of Galois extensions of fields and rings, strongly graded algebras, and affine algebraic principal homogeneous spaces.

In the literature on Hopf-Galois extensions (see [3] and the literature cited there), it is common to make the assumption that H is a Hopf algebra. An exception is [1], where cleft extensions over a bialgebra are considered. The question whether it is possible at all for a bialgebra H which does not have an antipode to admit Hopf-Galois extensions was brought to the attention of the author by Yukio Doi in connection with [4].

As the main result of this paper we will show that any k-bialgebra that admits a k-faithfully flat Hopf-Galois extension is a Hopf algebra. The proof uses a technical lemma (a descent argument) that might be of use in other situations.

By [1, Prop. 5, (3)] a bialgebra H has to have an antipode if there is a cleft H-comodule algebra A which is an augmented k-algebra. Since cleft extensions are Galois by [1, Thm. 9], this is a special case of our Theorem, if A is faithfully flat. In fact, if A is H-cleft then by [1, Prop. 5, (1)] at least the natural map $H \to A \otimes H$ is convolution invertible, and from that and our Lemma it follows that the identity on H is also, provided A is k-faithfully flat.

To work with the inverse of the Galois map κ we use the notation $\kappa^{-1}(1 \otimes h) =: \sum \ell_i(h) \otimes r_i(h)$ for $h \in H$. By definition we have

$$\sum \ell_i(h) r_i(h)_{(0)} \otimes r_i(h)_{(1)} = 1 \otimes h. \tag{1}$$

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We will need the following three equations:

$$\sum \ell_i(h)\otimes r_i(h)_{(0)}\otimes r_i(h)_{(1)} = \sum \ell_i(h_{(1)})\otimes r_i(h_{(1)})\otimes h_{(2)}, \tag{2}$$

$$\sum a_{(0)}\ell_i(a_{(1)})\otimes r_i(a_{(1)}) = 1\otimes a, \tag{3}$$

$$\sum \ell_i(h)r_i(h) = \varepsilon(h)1_A. \tag{4}$$

All of them are contained in a longer list of equations in [5, Rem. 3.4], and we only prove them for the sake of completeness (and since [5] is written under the general assumption that H is a Hopf algebra). It is enough to prove (2) after applying the isomorphism $\kappa\otimes \mathrm{id}_H$. We have

$$\begin{aligned}(\kappa\otimes\mathrm{id}_H)(\sum \ell_i(h)\otimes r_i(h)_{(0)}\otimes r_i(h)_{(1)}) &= \sum \ell_i(h)r_i(h)_{(0)}\otimes r_i(h)_{(1)}\otimes r_i(h)_{(2)}\\ = \sum 1\otimes h_{(1)}\otimes h_{(2)} &= \sum \ell_i(h_{(1)})r_i(h_{(1)})_{(0)}\otimes r_i(h_{(1)})_{(1)}\otimes h_{(2)}\\ &= \sum(\kappa\otimes\mathrm{id}_H)(\sum \ell_i(h_{(1)})\otimes r_i(h_{(1)})\otimes h_{(2)}),\end{aligned}$$

where both the second and third equality are applications of (1).

For (3) note that κ is obviously left A-linear if we equip both domain and codomain with the A-module structure induced by the left tensorand. Left A-linearity of κ^{-1} yields

$$1\otimes a = \kappa^{-1}(a_{(0)}\otimes a_{(1)}) = a_{(0)}\kappa^{-1}(1\otimes a_{(1)}) = \sum a_{(0)}\ell_i(a_{(1)})\otimes r_i(a_{(1)}).$$

Equation (4) follows from $(A\otimes\varepsilon)\kappa(x\otimes y) = xy$.

Theorem. *Let H be a k-bialgebra and A a right H-Galois extension of $B := A^{\mathrm{co}\,H}$ such that A is faithfully flat over k. Then H is a Hopf algebra.*

Proof. Define $\hat{S}: H\to A\otimes H$ by $\hat{S}(h) = \sum \ell_i(h)_{(0)}r_i(h)\otimes \ell_i(h)_{(1)}$. We claim that $\hat{S}$ is a convolution inverse for the map $\eta_0 : H\ni h\mapsto 1\otimes h\in A\otimes H$. This means that we have to verify:

$$\sum \ell_i(h_{(1)})_{(0)}r_i(h_{(1)})\otimes \ell_i(h_{(1)})_{(1)}h_{(2)} = \varepsilon(h)1_A\otimes 1_H, \tag{5}$$

$$\sum \ell_i(h_{(2)})_{(0)}r_i(h_{(2)})\otimes h_{(1)}\ell_i(h_{(2)})_{(1)} = \varepsilon(h)1_A\otimes 1_H, \tag{6}$$

for all $h\in H$. For (5), we apply the map $A\otimes A\otimes H\ni x\otimes y\otimes g\mapsto x_{(0)}y\otimes x_{(1)}g\in A\otimes H$ to both sides of (2) to find

$$\begin{aligned}\sum \ell_i(h_{(1)})_{(0)}r_i(h_{(1)})\otimes \ell_i(h_{(1)})_{(1)}h_{(2)} &= \sum \ell_i(h)_{(0)}r_i(h)_{(0)}\otimes \ell_i(h)_{(1)}r_i(h)_{(1)}\\ &= \sum(\ell_i(h)r_i(h))_{(0)}\otimes(\ell_i(h)r_i(h))_{(1)} = \varepsilon(h)1\otimes 1,\end{aligned}$$

the last equality being (4).

For (6) we use the π-method of [2]; recall that for any left A-module M, we have a bijection

$$\pi : \mathrm{Hom}(H, M)\to \mathrm{Hom}_B(A, M)$$

given by $\pi(f)(a) = a_{(0)}f(a_{(1)})$. We apply this for $M = A\otimes H$ to the maps

$$\begin{aligned}f &: H\ni h\mapsto \sum \ell_i(h_{(2)})_{(0)}r_i(h_{(2)})\otimes h_{(1)}\ell_i(h_{(2)})_{(1)}\in A\otimes H,\\ g &: H\ni h\mapsto \varepsilon(h)1\otimes 1\in A\otimes H,\end{aligned}$$

and find

$$\pi(f)(a) = \sum a_{(0)}\ell_i(a_{(2)})_{(0)}r_i(a_{(2)}) \otimes a_{(1)}\ell_i(a_{(2)})_{(1)}$$
$$= \sum (a_{(0)}\ell_i(a_{(1)}))_{(0)}r_i(a_{(1)}) \otimes (a_{(0)}\ell_i(a_{(1)}))_{(1)} = a \otimes 1 = \pi(g)(a),$$

where the third equality results from applying the map $A \otimes A \ni x \otimes y \mapsto x_{(0)}y \otimes x_{(1)} \in A \otimes H$ to both sides of (3). We conclude $f = g$.

Once we have found a convolution inverse $\hat{S}$ for η_0, the proof of our theorem is a direct application of the following lemma. □

Lemma. *Let C be a k-coalgebra and H a k-algebra. Let $f : C \to H$ be a map. If there is a k-faithfully flat k-algebra A such that the map $\hat{f} : C \ni c \mapsto 1 \otimes f(c) \in A \otimes H$ is convolution invertible, then f is convolution invertible.*

Proof. Let $\hat{g}$ be a convolution inverse for $\hat{f}$. By faithful flatness,

$$0 \to H \xrightarrow{\eta_0} A \otimes H \underset{\eta_2}{\overset{\eta_1}{\rightrightarrows}} A \otimes A \otimes H$$

is an equalizer, where $\eta_0(h) = 1 \otimes h$, $\eta_1(a \otimes h) = 1 \otimes a \otimes h$ and $\eta_2(a \otimes h) = a \otimes 1 \otimes h$. Since η_1 is an algebra map and $\hat{g}$ is a convolution inverse for $\hat{f}$, the map $\eta_1\hat{g} : C \to A \otimes A \otimes H$ is a convolution inverse for $\eta_1\hat{f}$. Similarly, $\eta_2\hat{g}$ is a convolution inverse for $\eta_2\hat{f}$. Since $\eta_1\hat{f} = \eta_2\hat{f}$, it follows that $\eta_1\hat{g} = \eta_2\hat{g}$. The universal property of the equalizer implies that there is a unique map $g : C \to H$ satisfying $\hat{g} = \eta_0 g$, that is, $\hat{g}(h) = 1_A \otimes g(h)$. Since $\hat{g}$ is a left convolution inverse for $\hat{f}$, we have $\epsilon(h)1_A \otimes 1_H = \hat{g}(h_{(1)})\hat{f}(h_{(2)}) = 1_A \otimes g(h_{(1)})f(h_{(2)})$. Now injectivity of η_0 implies that g is a left convolution inverse for f. Similarly g is a right convolution inverse for f. □

While the Lemma seems to be a fairly straightforward application of faithfully flat descent, we have not found it in the literature. Note that the only part of faithful flatness used in this paper is exactness of the equalizer diagram in the proof of the lemma. This holds of course if tensoring with A only *reflects* exactness; A does not have to be flat.

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