PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 1, January 1997, Pages 255–258
S 0002-9939(97)03721-0

A GENERALIZATION OF THE CLASSICAL SPHERE THEOREM

CHANGYU XIA

(Communicated by Christopher Croke)

Abstract. In this paper, we prove a sphere theorem for Riemannian manifolds with partially positive curvature which generalizes the classical sphere theorem.

1. Introduction

The celebrated Rauch-Berger-Klingenberg sphere theorem [1] states that, if M is an n-dimensional complete simply connected Riemannian manifold with sectional curvature K_M satisfying $1 < K_M \leq 4$, then M is homeomorphic to an n-sphere S^n. In 1977, Grove and Shiohama [2] generalized this sphere theorem to the following form : If the sectional curvature and diameter of M satisfy $d(M) > \frac{\pi}{2}$ and $K_M \geq 1$ then M is homeomorphic to S^n. In this paper, we shall prove a sphere theorem for manifolds with partially positive curvature which also generalizes the Rauch-Berger-Klingenberg sphere theorem.

Before stating our results, we fix some notations for the k-th Ricci curvature of a Riemannian manifold (Cf. [4], [6]). Let M be an n-dimensional Riemannian manifold. If, for any point $x \in M$ and any $(k+1)$-mutually orthogonal unit tangent vectors $e, e_1, ..., e_k \in T_xM$, we have $\sum_{i=1}^{k} K(e \wedge e_i) \geq kc$ (resp., $> kc$), we say that the k-th Ricci curvature of M is not less than kc (resp., larger than kc) and denote this fact by $\mathrm{Ric}_{(k)}(M) \geq kc$ (resp., $> kc$). Here, $K(e \wedge e_i)$ denotes the sectional curvature of the plane spanned by e and e_i $(1 \leq i \leq k)$. Thus, $\mathrm{Ric}_{(1)}(M) \geq c$ (resp., $> c$) is equivalent to $K_M \geq c$ (resp., $> c$) and $\mathrm{Ric}_{(n-1)}(M) \geq (n-1)c$ (resp., $> (n-1)c$) is equivalent to the fact that the Ricci curvature of M satisfies $\mathrm{Ric}_M \geq (n-1)c$ (resp., $> (n-1)c$). Also, it is easy to see that $\mathrm{Ric}_{(k)}(M) \geq kc$ (resp., $> kc$) implies $\mathrm{Ric}_{(m)}(M) \geq mc$ (resp., $> mc$) for $m \geq k$.

Now we can state our main theorem as follows.

Theorem 1. *Let M be an $n(\geq 3)$-dimensinal complete simply connected Riemannian manifold with $\mathrm{Ric}_{([\frac{n+1}{2}])}(M) > [\frac{n+1}{2}]$. If there exists a point $p \in M$ such that the injectivity radius i_p of M at p satisfies $i_p \geq \frac{\pi}{2}$, then M is homeomorphic to an n-sphere S^n.*

Received by the editors August 2, 1995.
1991 *Mathematics Subject Classification.* Primary 53C20.
Key words and phrases. Curvature, sphere theorem.
This work was partially supported by the JSPS postdoctoral fellowship and National Natural Science Foundation of China.

Let $M = CP^2(4)$ be the complex projective space of complex dimension 2 and of holomorphic sectional curvature 4. Then the injectivity radius of M is $\frac{\pi}{2}$ and $\mathrm{Ric}_{(3)}(M) = 6 > 3$, but M is not homeomorphic to S^4. This example shows that the curvature condition " $\mathrm{Ric}_{([\frac{n+1}{2}])}(M) > [\frac{n+1}{2}]$ " in Theorem 1 can't be weakened to " $\mathrm{Ric}_{(1+[\frac{n+1}{2}])}(M) > 1 + [\frac{n+1}{2}]$ ".

We remark that when $n = 3$, Theorem 1 is a special case of Hamilton's theorem [3]. An important result of Hartman [4] states that if M is an $n(\geq 3)$-dimensional complete simply connected Riemannian manifold with $K_M \leq 4$ and $\mathrm{Ric}_{(n-2)}(M) \geq n-2$, then the injectivity radius of M satisfies $i(M) \geq \frac{\pi}{2}$. Thus, Theorem 1 immediately implies the following generalized form of the classical sphere theorem:

Theorem 2. *Let M be an $n(\geq 3)$-dimensional complete simply connected Riemannian manifold. If $K_M \leq 4$ and $\mathrm{Ric}_{([\frac{n+1}{2}])}(M) > [\frac{n+1}{2}]$, then M is homeomorphic to S^n.*

2. A Proof of Theorem 1

Before proving Theorem 1, we list some basic facts we need. Let M be a compact n-dimensional Riemannian manifold with boundary. For any $1 \leq k \leq n-1$, let $\Lambda_k : \partial M \to R$ be the function on ∂M defined by $\Lambda_k(p)=$ the minimum of all sums of any k eigenvalues of the second fundamental form of ∂M at p with respect to the outward-pointing unit normal N at p. ∂M is called k-convex if $\Lambda_k > 0$ on ∂M.

Lemma 1 ([6])**.** *Let M be an n-dimensional compact Riemannian manifold with boundary ∂M and let k be an integer satisfying $1 \leq k \leq n-1$. If $\mathrm{Ric}_{(k)}(M) > 0$ and M has k-convex boundary, then M has the homotopy type of a CW complex obtained from ∂M by attaching a finite number of cells each of dimension $\geq n-k+1$.*

Proof of Theorem 1. When $n = 3$, Theorem 1 follows from Hamilton's theorem [3]. Thus, we can assume $n \geq 4$. Let $B(p, r)$ be the open geodesic ball of radius r with center p. Since the injectivity radius of M at p is not less than $\frac{\pi}{2}$, thus, for any $r \in (0, \frac{\pi}{2})$, $B(p, r)$ is homeomorphic to an n-disk and the boundary $\partial B(p, r)$ of $B(p, r)$ is a smooth hypersurface and it is homeomorphic to an $(n-1)$-sphere S^{n-1}. From $\mathrm{Ric}_{([\frac{n+1}{2}])}(M) > [\frac{n+1}{2}]$, we know that $\mathrm{Ric}_M > n-1$, thus M is compact by Bonnet-Myers theorem [1], and so we can find a positive constant δ such that $\mathrm{Ric}_{([\frac{n+1}{2}])}(M) > [\frac{n+1}{2}] + \delta$. Take $l = \frac{\pi}{2\sqrt{1+\frac{\delta}{[\frac{n+1}{2}]}}}$; then $l < \frac{\pi}{2}$. Now, we want to show that $M - B(p, l)$ has $[\frac{n+1}{2}]$-convex boundary. To do this, we denote by ∇ the Riemannian connection of $M - B(p, l)$ and S the second fundamental form of $M - B(p, l)$ with respect to the outward-pointing unit normal N, i.e., $S(u, v) = \langle \nabla_u N, v\rangle$ for $u,\ v \in T\partial B(p, l)$. Let q be any point on the boundary of $M - B(p, l)$ and $\gamma :\ [0,\ l] \to \overline{B}(p, l)$ be a normal geodesic from p to q that realizes the minimum distance between them. An argument of the first variation of arc-length shows that γ strikes $\partial B(p, l)$ orthogonally. For any k orthonormal vectors $e_1, ..., e_k$ in $T_q\partial B(p, l)$, we denote by $E_i(s)$ the parallel translate of e_i to $\gamma(s)$ along γ and define $W_i(s) = \sin\frac{\pi s}{2l} E_i(s),\ i = 1, ..., k$. Each vector field W_i gives rise to a variation with length $L_{W_i}(t)(\ t \in (-\epsilon, \epsilon)\ )$ of the variational curves of the geodesic γ keeping one end point p fixed and other end points on $\partial B(p, l)$. The first variation of the arc-length $L'_{W_i}(0) = 0$, and the second variation of the arc-length $L''_{W_i}(0) \geq 0, i = 1, ..., k$, since γ is a minimizing geodesic from p to q. It then follows from the second variation formula of arc-length (see p.99 in [5]) that for $i = 1, ..., k$,

$$
\begin{aligned}
(1) \qquad 0 &\leq L''_{W_i}(0) \\
&= \langle \nabla_{e_i} e_i, \gamma'(l)\rangle + \int_0^l \{|\nabla_{\gamma'(s)} W_i(s)|^2 \\
&\quad -\langle R(\gamma'(s), W_i(s))W_i(s), \gamma'(s)\rangle\} ds \\
&= S(e_i, e_i) + \int_0^l \{|\nabla_{\gamma'(s)} W_i(s)|^2 \\
&\quad -\langle R(\gamma'(s), W_i(s))W_i(s), \gamma'(s)\rangle\} ds,
\end{aligned}
$$

where R is the curvature tensor of M.

Thus, from $\mathrm{Ric}_{([\frac{n+1}{2}])} > [\frac{n+1}{2}] + \delta$ and (1), we get

$$
\begin{aligned}
(2) \qquad \sum_{i=1}^{[\frac{n+1}{2}]} S(e_i, e_i) &\geq -\sum_{i=1}^{[\frac{n+1}{2}]} \int_0^l \{|\nabla_{\gamma'(s)} W_i(s)|^2 \\
&\quad -\langle R(\gamma'(s), W_i(s))W_i(s), \gamma'(s)\rangle\} ds \\
&= -\sum_{i=1}^{[\frac{n+1}{2}]} \int_0^l \{\frac{\pi^2}{4l^2} \cos^2 \frac{\pi s}{2l} \\
&\quad -\sin^2 \frac{\pi s}{2l} \langle R(\gamma'(s), E_i(s))E_i(s), \gamma'(s)\rangle\} ds \\
&> -[\frac{n+1}{2}] \int_0^l (\frac{\pi^2}{4l^2} \cos^2 \frac{\pi s}{2l} - (1 + \frac{\delta}{[\frac{n+1}{2}]}) \sin^2 \frac{\pi s}{2l}) ds \\
&= -[\frac{n+1}{2}] \cdot \frac{2l}{\pi} \int_0^{\frac{\pi}{2}} (\frac{\pi^2}{4l^2} \cos^2 s - (1 + \frac{\delta}{[\frac{n+1}{2}]}) \sin^2 s) ds \\
&= -[\frac{n+1}{2}] \cdot \frac{l}{2} (\frac{\pi^2}{4l^2} - (1 + \frac{\delta}{[\frac{n+1}{2}]})) \\
&= 0.
\end{aligned}
$$

Therefore, $M - B(p, l)$ has $[\frac{n+1}{2}] = n - [\frac{n}{2}]$-convex boundary. From Lemma 1 we conclude that $M - B(p, l)$ has the homotopy type of a CW complex obtained from $\partial B(p, l) \simeq S^{n-1}$ by attaching a finite number of cells each of dimension $\geq [\frac{n}{2}] + 1$. So does M since $M = B(p, l) \cup (M - B(p, l))$ and $B(p, l) \simeq D^n$, the n-disk. Thus, $\pi_i(M, B(p, l)) = 0$ for $1 \leq i \leq [\frac{n}{2}]$. From the homotopy exact sequence we know that $\pi_i(M) = 0$ for $1 \leq i \leq [\frac{n}{2}]$. By a basic argument in topology, one can deduce that M is homeomorphic to the sphere S^n. This completes the proof of Theorem 1. □

References

[1] J. Cheeger and D. Ebin, Comparison Theorems in Riemannian Geometry, North-holland, New-York, 1975.

[2] K. Grove and K. Shiohama, A generalized sphere theorem, Ann. of Math., **106** (1977), 201-211. MR **58:**18268

[3] R. Hamilton, Three manifolds with positive Ricci curvature, J. Differential Geom., **17** (1982), 255-306. MR **84a:**53050

[4] P. Hartman, Oscillation criteria for self-adjoint second-order differential systems and " principal sectional curvature " , J. Differential Equations **34** (1979), 326-338. MR **81a:**34034

[5] W. Klingenberg, Riemannian Geometry, Berlin, New York : de Gruyter, 1982. MR **84j:**53001

[6] H. Wu, Manifolds of partially positive curvature, Indiana Univ. Math. J., **36** (1987), 525-548. MR **88k:**53068

[7] C.Y. Xia, Rigidity and sphere theorem for manifolds with positive Ricci curvature, manuscripta math., **85** (1994), 79-87. MR **95j:**53057

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China

Current address: Instituto de Matemática Pure e Aplicada, Estrada Dona Castorina, 110, 22460-320 Rio de Janeiro RJ, Brasil

E-mail address: `xiacy@impa.br`