PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 1, January 1997, Pages 163–166
S 0002-9939(97)03743-X

FREDHOLM COMPOSITION OPERATORS

BARBARA D. MACCLUER

(Communicated by Palle E. T. Jorgensen)

Abstract. Fredholm composition operators on a variety of Hilbert spaces of analytic functions on domains in $C^N, N \geq 1$, are characterized.

For Ω a domain in C^N, $N \geq 1$, and φ an analytic map of Ω into Ω, the composition operator C_φ is defined by $C_\varphi(f) = f \circ \varphi$, where f is analytic on Ω. We consider C_φ acting on various Hilbert spaces $\mathcal{H}$ of analytic functions on Ω. It is known that under quite general and natural conditions on $\mathcal{H}$ (see Theorem 1.6 of [CM95]), if C_φ is a bounded invertible operator on $\mathcal{H}$, then φ must be an automorphism of Ω; that is, a one-to-one map of Ω onto Ω. The converse holds trivially if $\mathcal{H}$ is an automorphism-invariant space. Various authors have considered the question of when C_φ is Fredholm, that is, invertible modulo the compact operators, for particular choices of $\mathcal{H}$ (e.g. on the Hardy space $H^2(D)$ [CiTW74], the Dirichlet space $\mathcal{D}(D)$ [Cim77], spaces containing the Dirichlet space [CM95], Hardy and Bergman spaces in the disk [Bou90], weighted Dirichlet spaces in the disk [JM95], and various spaces on domains in C^N [Hat94]). In all of these settings the same result holds: if C_φ is Fredholm, then φ must be an automorphism of the domain Ω. The purpose of this note is to give a simple proof of this, applicable to a wide variety of Hilbert spaces in both one and several variables.

A Hilbert space $\mathcal{H}$ of analytic functions on the disk D is called a weighted Hardy space if the monomials are a complete orthogonal set of non-zero vectors in $\mathcal{H}$. Setting $\beta(n) = \|z^n\|$ we have $\|f\|^2 = \sum_{n=0}^\infty |a_n|^2 \beta(n)^2$ where $f = \sum_{n=0}^\infty a_n z^n$ is in $\mathcal{H}$. We denote the weighted Hardy space with weight sequence $\{\beta(n)\}$ by $H^2(\beta)$. The choices $\beta(n)^2 = (n+1)^a$ for appropriate a give the classical Hardy space, the standard (weighted) Bergman spaces and (weighted) Dirichlet spaces. The reproducing kernel function for evaluation at w in $H^2(\beta)$ is

$$K_w(z) = \sum_{n=0}^\infty \frac{z^n \overline{w}^n}{\beta(n)^2}$$

and the kernel for evaluation of the k^{th} derivative at w is

$$K_w^{(k)}(z) \equiv \sum_{n=k}^\infty \frac{n!}{(n-k)!} \frac{z^n \overline{w}^{n-k}}{\beta(n)^2}.$$

A more detailed discussion of weighted Hardy spaces can be found in Section 2.1 of [CM95].

Received by the editors July 3, 1995.
1991 *Mathematics Subject Classification.* Primary 47B38.
Supported in part by National Science Foundation Grant DMS-9300525.

Lemma 1. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on a domain Ω in C^N. If there exists a sequence of functions f_k in $\mathcal{H}$ with $\|f_k\| = 1$ and $f_k \to 0$ weakly as $k \to \infty$ such that $\|C_\varphi^*(f_k)\| \to 0$, then C_φ is not Fredholm on $\mathcal{H}$.*

Proof. Suppose S is any bounded operator on $\mathcal{H}$. The hypotheses will guarantee that $\|SC_\varphi^*(f_k)\| \to 0$ but $\|(I+Q)f_k\| \to 1$ as $k \to \infty$ for every compact operator Q. From this it follows that $SC_\varphi^* - I$ cannot be compact. Thus C_φ^*, and hence C_φ, is not Fredholm. □

In the next theorem we will apply the lemma with the sequence f_k chosen to be a suitable sequence of normalized reproducing kernel functions (in general, for evaluation of derivatives). We state the theorem first for weighted Hardy spaces in the disk; however, as we will see, the argument will carry over with no new ideas to analogous spaces in the ball $B_N, N > 1$, as well as other domains in C^N.

Theorem 2. *Let $H^2(\beta)$ be a weighted Hardy space in the disk such that*

$$\sum_{n=0}^{\infty} \frac{n^{2j}}{\beta(n)^2} = \infty \tag{1}$$

for some non-negative integer j. If C_φ is Fredholm on $H^2(\beta)$, then φ is an automorphism of the disk.

Proof. It is well known that on any reproducing kernel Hilbert space of analytic functions that contains the polynomials, φ must be univalent if C_φ is Fredholm ([Bou90],[CM95]). (Roughly speaking, if φ is not univalent, then the kernel of C_φ^* will contain an infinite linearly independent set whose elements are differences of reproducing kernel functions.) Thus we need only show that φ maps D *onto* D. If not, find $v \in \partial\varphi(D) \cap D$ and $z_k \in D$ such that $\varphi(z_k) \to v$. By the open mapping theorem we must have $|z_k| \to 1$.

Let j be the least non-negative integer so that the sum in Equation (1) diverges. If $j = 0$, set $f_k = K_{z_k}/\|K_{z_k}\|$ where K_w denotes the kernel function for evaluation at w in $H^2(\beta)$. Then $\|f_k\| = 1$ and $f_k \to 0$ weakly as $k \to \infty$, since $\|K_{z_k}\| \to \infty$ (see Theorem 2.17 in [CM95]). We have

$$\|C_\varphi^*(f_k)\| = \frac{\|K_{\varphi(z_k)}\|}{\|K_{z_k}\|} \to 0$$

since $\|K_{\varphi(z_k)}\| \to \|K_v\| < \infty$ while $\|K_{z_k}\| \to \infty$. By Lemma 1, C_φ is not Fredholm.

If $j > 0$, let $f_k = K_{z_k}^{(j)}/\|K_{z_k}^{(j)}\|$ be the normalized reproducing kernel function for evaluation of the j^{th} derivative at z_k. By Proposition 7.13 in [CM95] $f_k \to 0$ weakly as $k \to \infty$. (Again the crucial step is that divergence of the sum in Equation (1) implies $\|K_w^{(j)}\| \to \infty$ as $|w| \to 1$.) A straightforward computation gives the following formulae: If $j = 1$,

$$C_\varphi^*(K_w^{(1)}) = \overline{\varphi'(w)} K_{\varphi(w)}^{(1)}$$

and if $j > 1$,

$$C_\varphi^*(K_w^{(j)}) = \overline{[\varphi'(w)]^j} K_{\varphi(w)}^{(j)} + \overline{\varphi^{(j)}(w)} K_{\varphi(w)}^{(1)} + \text{ lower order terms}$$

where the "lower order terms" involve kernels for derivatives of order less than j at $\varphi(w)$ with coefficients involving products of derivatives of φ at w of order less than j. From this it will follow that $\|C_\varphi^*(f_k)\| \to 0$ as $k \to \infty$. To see this we first remark that since j is the least integer for which $\sum_{n=0}^{\infty} \frac{n^{2j}}{\beta(n)^2} = \infty$, the norms of the kernel functions for evaluation of derivatives of order less than j remain bounded in D, and hence so do all derivatives of φ of order less than j. Since $\varphi(z_k) \to v$ where v is in D, the only term in question in evaluating $\|C_\varphi^*(f_k)\|$ is

$$\frac{\|\overline{\varphi^{(j)}(z_k)} K^{(1)}_{\varphi(z_k)}\|}{\|K^{(j)}_{z_k}\|} = \left|\langle \varphi, K^{(j)}_{z_k}/\|K^{(j)}_{z_k}\|\rangle\right| \|K^{(1)}_{\varphi(z_k)}\|$$

which tends to 0 by the weak convergence of $K^{(j)}_{z_k}/\|K^{(j)}_{z_k}\|$ to 0 as $k \to \infty$. By the lemma, C_φ is not Fredholm. ☐

The theorem extends easily to weighted Hardy spaces $H^2(\beta, B_N)$ in the ball $B_N, N > 1$. By a weighted Hardy space in B_N we mean a Hilbert space of analytic functions in B_N for which the monomials z^α, as α ranges over all multi-indices, form a complete orthogonal set of non-zero vectors with

$$\frac{\|z^{\alpha_1}\|}{\|z^{\alpha_1}\|_2} = \frac{\|z^{\alpha_2}\|}{\|z^{\alpha_2}\|_2}$$

whenever the total orders of α_1 and α_2 are equal. Here $\|\cdot\|_2$ denotes the norm in $L^2(\sigma_N)$ where σ_N is a normalized Lebesgue measure on ∂B_N. Defining $\beta(s)$ to be $\|z^\alpha\|/\|z^\alpha\|_2$ for α any multi-index of total order s we have $\|f\|^2 = \sum_{s=0}^{\infty} \|f_s\|_2^2 \beta(s)^2$ where $f = \sum_{s=0}^{\infty} f_s$ is the homogeneous expansion of $f \in \mathcal{H}$. As was the case in one variable, the usual Hardy space $H^2(B_N)$, the standard weighted Bergman spaces, and weighted Dirichlet spaces can all be described as weighted Hardy spaces for appropriate choice of the sequence $\beta(s)$. For all of these classical spaces the weight sequence $\beta(s)$ will satisfy the hypothesis of the next result. The kernel function in $H^2(\beta, B_N)$ for evaluation at w is

$$K_w(z) = \sum_{s=0}^{\infty} \frac{(N-1+s)!}{(N-1)!s!} \frac{1}{\beta(s)^2} \sum_{|\alpha|=s} \frac{z^\alpha \overline{w}^\alpha}{\alpha!} s! = \sum_{s=0}^{\infty} \frac{(N-1+s)!}{(N-1)!s!} \frac{1}{\beta(s)^2} \langle z, w\rangle^s$$

and the kernel K^γ for evaluation of the derivative $D^\gamma, \gamma = (\gamma_1, \dots, \gamma_N)$, is obtained as

$$\frac{\partial^{|\gamma|}}{\partial \overline{w_1}^{\gamma_1} \cdots \partial \overline{w_N}^{\gamma_N}} K_w(z)$$

where $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_N$. (See Section 2.1 of [CM95] for further details.)

Theorem 3. *Let $H^2(\beta) \equiv H^2(\beta, B_N)$ be a weighted Hardy space in B_N such that*

$$\sum_{s=0}^{\infty} \frac{s^J}{\beta(s)^2} = \infty \tag{2}$$

for some non-negative integer J. If C_φ is Fredholm on $H^2(\beta)$, then φ must be an automorphism of B_N.

Proof. The idea of the proof is exactly the same as in the $N = 1$ case so we will just outline it. As before φ must be univalent (see comments following Lemma 3.26 in [CM95]), so only surjectivity is in question. If $\varphi(B_N)$ is properly contained in B_N, we will have a sequence $\{z_k\}$ in B_N tending to a point of ∂B_N but $\varphi(z_k) \to v \in B_N$.

Let J be the least non-negative integer for which Equation 2 holds. If $J \leq N-1$, then $\|K_w\| \to \infty$ as $|w| \to 1$ and the argument is identical to that in the case $N = 1$.

If $J > N-1$, then straightforward estimates on the kernel functions for derivatives show that we may find a positive integer j such that

- For each multi-index α with $|\alpha| \le j-1, \|K_w^\alpha\|$ remains bounded as $|w| \to 1$, and hence $D^\alpha g(w)$ remains bounded as $|w| \to 1$, for every $g \in H^2(\beta)$.
- There exists a multi-index γ of total order j such that $\|K_w^\gamma\| \to \infty$ as $|w| \to 1$. For this γ, $K_w^\gamma/\|K_w^\gamma\|$ tends to 0 weakly as $|w| \to 1$.

A computation with the chain rule identifies $C_\varphi^*(K_{z_k}^\gamma)$ for our chosen sequence z_k. It will be a sum of terms most of which will be products of derivatives of total order less than $|\gamma|$ of the coordinate functions φ_m of φ at z_k with kernel functions for derivatives at $\varphi(z_k)$. The corresponding terms in $C_\varphi^*(K_{z_k}^\gamma/\|K_{z_k}^\gamma\|)$ tend to zero as $k \to \infty$. The remaining terms will be products of $\overline{D^\gamma \varphi_m(z_k)}$ with kernel functions for derivatives at $\varphi(z_k)$. In the expression for $C_\varphi^*(K_{z_k}^\gamma/\|K_{z_k}^\gamma\|)$ these terms give rise to terms of the form

$$\frac{\overline{D^\gamma \varphi_m(z_k)} K_{\varphi(z_k)}^\tau}{\|K_{z_k}^\gamma\|} = \langle \varphi_m, K_{z_k}^\gamma/\|K_{z_k}^\gamma\|\rangle K_{\varphi(z_k)}^\tau$$

whose norms go to zero as $k \to \infty$ by weak convergence, since each coordinate function φ_m is in $H^2(\beta)$. Thus φ must be an automorphism if C_φ is Fredholm. ☐

It is clear that this method can be applied to Hilbert spaces of analytic functions on more general domains in C^N. We close with just one such example.

Theorem 4. *Let Ω be a C^∞- bounded strongly pseudoconvex domain in C^N and let $\mathcal{H}$ be the Bergman space $A^2(\Omega)$ of holomorphic functions in $L^2(\Omega, dV)$ where dV is volume measure. If C_φ is Fredholm on $\mathcal{H}$, then φ is an automorphism of Ω.*

Proof. Let $K(z,w)$ be the Bergman kernel for Ω. Since $K(w,w) \to \infty$ as w approaches $\partial\Omega$ and functions in $A^2(\Omega)$ can be approximated in norm by functions holomorphic in a neighborhood of $\overline{\Omega}$ [K71] the normalized reproducing kernel functions $K(z,w)/K(w,w)$ tend to 0 weakly for any sequence of w's approaching $\partial\Omega$. If φ does not map Ω onto Ω, find $v \in \Omega$ and $z_j \in \Omega$ with $\varphi(z_j) \to v$ and $z_j \to \partial\Omega$. Then $f_j(z) = K(z,z_j)/K(z_j,z_j)$ satisfies the hypotheses of Lemma 1 and C_φ is not Fredholm. As before φ must be univalent, since otherwise C_φ^* has infinite dimensional kernel. ☐

References

[Bou90] P. S. Bourdon, Fredholm multiplication and composition operators on the Hardy space, J. Integral Equations Operator Theory **13** (1990) 607–610. MR **91m:**47038

[Cim77] J. A. Cima, A theorem on composition operators, Banach Spaces of Analytic Functions, Lecture Notes in Math., Vol. 604, Springer–Verlag, Berlin, 1977, 21–24. MR **57:**13562

[CiTW74] J. A. Cima, J. Thomson, and W. R. Wogen, On some properties of composition operators, Indiana Univ. Math. J. **24** (1974) 215–220. MR **50:**2979

[CM95] C. C. Cowen and B. D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, 1995.

[Hat94] O. Hatori, Fredholm composition operators on spaces of holomorphic functions, J. Integral Equations Operator Theory **18** (1994) 202–210. MR **95a:**47027

[JM95] M. Jovovic and B. D. MacCluer, Composition operators on Dirichlet spaces, preprint.

[K71] N. Kerzman, Holder and L^p estimates for solutions of $\overline{\partial} u = f$ in strongly pseudoconvex domains, Comm. Pure Appl. Math. **24** (1971) 301–379. MR **43:**7658

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903
E-mail address: `maccluer@virginia.edu`