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A METRIC CONDITION WHICH IMPLIES DIMENSION ≤ 1

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Abstract. A class of 1-dimensional spaces is distinguished by special type embeddings in compacta, or a corresponding metric property. In this setting, a simple proof of the Oversteegen-Tymchatyn theorem that the spaces of homeomorphisms of the Sierpiński's Carpet and the Menger Universal Curve have dimension ≤ 1 is given.

1. Introduction

We consider only separable metrizable spaces, and by a compactum we mean a compact metrizable space.

Definition 1.1. A subset X of a compactum K is L-embedded in K if for any open cover $\mathcal{U}$ of K there is a neighbourhood U of X in K such that every continuum in U is contained in some element of $\mathcal{U}$.

Theorem 1.2. *L-embedded subsets of compacta are at most 1-dimensional.*

We prove this theorem in sec. 3, where we consider a metric condition (L), related to L-embeddings. Let us notice that a non-trivial connected set can be L-embedded in a compactum, cf. sec. 4.

There is a link between L-embeddings and a notion of almost 0-dimensionality introduced in [4].

Definition 1.3 (Oversteegen - Tymchatyn [4])**.** A space X is almost 0-dimensional provided there is a countable basis $\mathcal{B}$ in X such that for each pair G,H of elements of $\mathcal{B}$ with disjoint closures there is an open-and-closed set W with $G \subset W \subset X \setminus H$.

Theorem 1.4 (Oversteegen - Tymchatyn [4])**.** *Almost 0-dimensional spaces are at most 1-dimensional.*

The spaces of homeomorphisms of the Sierpiński's Carpet, the Menger Universal Curve, or more generally, the Menger compacta M_n^k with $k > n$, are almost 0-dimensional; see Oversteegen and Tymchatyn [4], Th.5, cf. also [1], Th.1.3. Theorem 1.4 provided the first proof that the homeomorphism spaces are 1-dimensional (the inequality $\geq$ was established earlier by Brechner [1], Corollary 3.1.1 and 3.2.1, cf. [1], Question 1 on page 533).

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However, the proof of Theorem 1.4 given in [4], based on a notion of R-trees, is rather complicated, and one of our objectives is to give a simple proof of this very interesting result. To this end, one can show that any almost 0-dimensional space can be L-embedded in some compactum, and then apply Theorem 1.2. We decided to include also, in sec. 2, an even more direct proof of the Oversteegen - Tymchatyn theorem, where L-embedding is used only implicitly, bypassing some difficulties in Theorem 1.2.

2. A proof of the Oversteegen-Tymchatyn theorem

Let X be an almost 0-dimensional space, and let $(A_0, B_0), (A_1, B_1)$ be two pairs of disjoint closed sets in X. We have to find partitions L_i in X between A_i and B_i with $L_0 \cap L_1 = \emptyset$; cf. [2], 1.7.9.

Let $\mathcal{B}$ be a countable basis described in Definition 1.3. For each pair G, H in $\mathcal{B}$ with disjoint closures fix a continuous map from X to $\{0, 1\}$ taking G to 0 and H to 1, and arrange the mappings into a sequence $f_1, f_2, \ldots$. Let $g_i : X \longrightarrow [0, 1]$ be continuous maps with $A_i = g_i^{-1}(0), B_i = g_i^{-1}(1)$, $i = 0, 1$, and finally, let ρ be a totally bounded metric for X.

Let us consider a totally bounded metric d on X, defined by

$$d(x, y) = \rho(x, y) + \sum_{i=0}^{1} \mid g_i(x) - g_i(y) \mid + \sum_{i=1}^{\infty} 2^{-i} \mid f_i(x) - f_i(y) \mid .$$

Let K be the compact completion of X with respect to d. (At this point, one could check, ignoring g_i, that X is L-embedded in K. We choose a more direct argument, although L-embedding will be hidden in our reasoning.)

For each $G \in \mathcal{B}$ choose G^* open in K with $G^* \cap X = G$, and let

$$U = \bigcup \{G^* : G \in \mathcal{B},\ \ \operatorname{diam} G^* < 1/3\},$$

where the diameter diam refers to the extension of the metric d over K.

If a continuum C in K intersects both sets G^* and H^*, their closures must meet, for otherwise the extension over K of a function f_i separating G and H would split C into two separate pieces. It follows that all continua in U have diameter < 1, and therefore no continuum in U joins the closures $\mathrm{cl}A_i$ and $\mathrm{cl}B_i$ of A_i and B_i in K, $i = 0, 1$.

Let $\varphi : K \longrightarrow [0, 1]$ be a continuous map with $U = \varphi^{-1}(0, 1]$ and let us consider the compact rings

$$K_n = \varphi^{-1}[1/(n + 1), 1/n],\ \ n = 1, 2, \ldots .$$

Then no component of K_n joins $\mathrm{cl}A_i$ and $\mathrm{cl}B_i$, and since both collections $\{K_{2n-i} : n = 1, 2, \ldots\}$, where $i = 0, 1$, are discrete in the space U, there are partitions S_i in U between $\mathrm{cl}A_i \cap U$ and $\mathrm{cl}B_i \cap U$ with $S_i \cap \bigcup\{K_{2n-1} : n = 1, 2, \ldots\} = \emptyset$. Then $S_0 \cap S_1 = \emptyset$, and $L_i = S_i \cap X$ are the partitions we were looking for.

3. A metric condition which implies dimension ≤ 1

In a metric space X, endowed with a metric ρ, we call a pair of sets A, B distant, provided

$$\inf\{\rho(a, b) : a \in A,\ b \in B\} > 0.$$

Definition 3.1. Given a pair A, B of disjoint sets in a metric space X we call a pair of open sets $G \supset A$, $H \supset B$ an L_ϵ-enlargement for the pair A, B if $G \cap H = \emptyset$ and $X \setminus (G \cup H)$ is a union of a discrete collection $\mathcal{F}$ of closed sets of diameter $\leq \epsilon$ such that for each $F \in \mathcal{F}$ the sets $\operatorname{cl} G \cap F$ and $\operatorname{cl} H \cap F$ are distant.

A metric space X has property (L) if for each pair of distant sets A, B in X and every $\epsilon > 0$, there is an L_ϵ-enlargement for the pair A, B.

Theorem 1.2 follows immediately from the following

Proposition 3.2. *Each L-embedded subspace of a compactum K has property (L) with respect to any metric inherited from K, and each separable metric space with property (L) is at most 1-dimensional.*

Proof. (A) Let X be L-embedded in a compactum K, let ρ be a metric on K, and let A, B be a pair of subsets of X with

$$\inf\{\rho(a, b) : a \in A,\, b \in B\} = \delta > 0.$$

Let $\epsilon > 0$. We have to find an L_ϵ-enlargement for the pair A, B. To this end, let us choose an open neighbourhood U of X in K such that all continua in U have diameter $\leq (1/2)\min(\epsilon, \delta)$. Let $\varphi : X \longrightarrow [0, 1]$ be a continuous map with $U = \varphi^{-1}(0, 1]$, and let $K_n = \varphi^{-1}[1/(n + 1), 1/n]$. In each compactum K_n, no component joins the closures $\mathrm{cl}A$ and $\mathrm{cl}B$ of A and B in K, and hence each K_n can be split into two disjoint closed sets containing $\mathrm{cl}A \cap K_n$ and $\mathrm{cl}B \cap K_n$, respectively. The collection of "even rings" K_{2n}, $n = 1, 2, \dots$, being discrete in U, one can find open sets V, W in U with $\mathrm{cl}A \cap U \subset V$, $\mathrm{cl}B \cap U \subset W$, $\mathrm{cl}V \cap \mathrm{cl}W \cap U = \emptyset$ and $\bigcup\{K_{2n} : n = 1, 2, \dots\} \subset V \cup W$. Then $U \setminus (V \cup W)$ is contained in the union of the collection of "odd rings" K_{2n-1}, $n = 1, 2, \dots$, which is discrete in the space U. Each K_{2n-1} is a compactum whose components have diameter $< \epsilon$, and therefore K_{2n-1} can be split into finitely many disjoint compacta of diameter $\leq \epsilon$. The compactness guarantees that the traces of $\mathrm{cl}V$ and $\mathrm{cl}W$ on any of these pieces are distant. Therefore, the pair $G = V \cap X$ and $H = W \cap X$ provides an L_ϵ-enlargement for A, B.

(B) Let X be a metric space with property (L). Let F be a closed set in X and $p \in X \setminus F$. We have to find open sets V, W in X with $p \in V$, $F \subset W$, and $\dim(X \setminus (V \cup W)) \leq 0$.

We shall define two increasing sequences of open sets

$$p \in G_1 \subset G_2 \subset \dots \quad , \quad F \subset H_1 \subset H_2 \subset \dots$$

such that $\mathrm{cl}G_n \cap \mathrm{cl}H_n = \emptyset$ and each pair G_{n+1}, H_{n+1} is an $L_{1/n}$-enlargement for the pair of the closures $\mathrm{cl}G_n$, $\mathrm{cl}H_n$.

We begin with a distant pair G_1, H_1 of open sets containing p and F, respectively. Suppose G_n, H_n have been defined. For n=1, we get G_2, H_2 directly from property (L).

Assume $n \geq 2$, and let $X \setminus (G_n \cup H_n)$ be the union of a discrete family $\mathcal{F}$ of closed sets such that for each $F \in \mathcal{F}$ the sets

$$A(F) = \operatorname{cl} G_n \cap F \quad , \quad B(F) = \operatorname{cl} H_n \cap F$$

are distant. Let $G(F) \supset A(F)$ and $H(F) \supset B(F)$ be an $L_{1/n}$-enlargement for the pair $A(F), B(F)$. For each $F \in \mathcal{F}$, let $U(F) \supset F$ be an open set such that the collection of closures $\{\text{cl}\, U(F) : F \in \mathcal{F}\}$ is discrete. Then we can define

$$G_{n+1} = G_n \cup \bigcup \{G(F) \cap U(F) : F \in \mathcal{F}\},$$

$$H_{n+1} = H_n \cup \bigcup \{H(F) \cap U(F) : F \in \mathcal{F}\}.$$

Finally, we put $V = \bigcup_n G_n$, $W = \bigcup_n H_n$. For each $\epsilon > 0$, the complement $X \setminus (V \cup W)$ is a union of a discrete collection of closed sets of diameter $\leq \epsilon$, and therefore it is at most 0-dimensional. □

4. Examples

4.1. Let X be a subspace of a compactum K such that all but countably many points of X have a basis of closed-and-open sets in K. Then, as one can easily check, X is L-embedded in K. In particular, a one-dimensional set defined by Kuratowski [3] is L-embedded in its closure in the plane.

We have already noticed (cf. the remark in brackets, following the definition of d in sec. 2) that each almost 0-dimensional space can be L-embedded in some compactum. The construction of Kuratowski we have just mentioned provides L-embedded sets which are not totally disconnected, and hence not almost 0-dimensional. But, as we shall see in the next example, non-trivial L-embedded sets may be even connected.

4.2. Let S be the space of the points in the separable Hilbert space l^2 with all coordinates rational. Then S is almost 0-dimensional [4], sec.1. Roberts [5] proved that one can add to S a single point p, obtaining a connected space $X = \{p\} \cup S$. We shall show that there is a compactification K of X in which X is L-embedded.

Let ρ be a totally bounded metric for X. Let $\mathcal{B}$ be a basis in S described in Definition 1.3 and let $f_i : S \longrightarrow \{-1, 1\}$ be continuous maps such that for each pair G, H of elements of $\mathcal{B}$ with disjoint closures in S, there is some f_i which takes G to -1 and H to 1. Let us consider continuous maps u_i on X defined by $u_i(p) = 0$ and $u_i(x) = \rho(x,p) \cdot f_i(x)$ for $x \in S$. Then

$$d(x, y) = \rho(x, y) + \sum_{i=1}^{\infty} 2^{-i} \mid u_i(x) - u_i(y) \mid$$

is a totally bounded metric on X. We shall verify that X is L-embedded in the compact completion K of X with respect to d.

The metric ρ and the functions u_i extend continuously over K, and we shall keep the same symbols for the extensions. Let

$$Z = \{x \in K : \rho(x, p) = 0\},$$

$$G_i = \{x \in K : u_i(x) < 0\}\ ,\ H_i = \{x \in K : u_i(x) > 0\}.$$

One readily checks that

$$(\star) \qquad K \setminus Z \subset G_i \cup H_i \quad \text{for} \ \ i = 1, 2, \ldots .$$

Let $\mathcal{U}$ be an open cover of K and let $\delta > 0$ be such that each set of diameter $\leq \delta$ in K is contained in some element of $\mathcal{U}$. For every $G \in \mathcal{B}$ choose $G^* \subset K \setminus Z$, open in K, with $G^* \cap X = G$, and let W be the union of the sets G^* of diameter $\leq \delta/16$. The neighbourhood U of X in K required by Definition 1.1, will be the union of W and the open $\delta/4$-ball $B(p, \delta/4)$ about p in K. Let us check that each continuum C

in U has diameter $\leq \delta$. Aiming at a contradiction, suppose that diam$C > \delta$. Then there is a continuum T in $C \cap W$ of diameter $\geq \delta/4$. Indeed, consider $q \in C$ with $d(p,q) > \delta/2$. If C is disjoint from $B(p, \delta/4)$, we can take $T = C$. Otherwise, let T be any continuum in C joining q with the boundary of $B(p, \delta/4)$. Let us consider $a, b \in T$ with $d(a,b) \geq \delta/4$. Since $T \subset W$, there are $G, H \in \mathcal{B}$ of diameter $\leq \delta/16$, with $a \in G^*$ and $b \in H^*$. But then the closures of G and H are disjoint and, for some i, u_i is negative on G and positive on H. By $(\star)$, T being disjoint from Z, the function u_i changes its sign, never vanishing on T, which contradicts connectivity of T.

References

1. B.Brechner, On the dimension of certain spaces of homeomorphisms, Trans.A.M.S. 121 (1966), 516-548. MR **32:**4662
2. R.Engelking, Dimension theory, Warszawa, 1978. MR **58:**2753b
3. K.Kuratowski, Une application des images de functions á la construction de certains ensembles singuliers, Mathematica 6 (1932), 120-123.
4. L.G.Oversteegen, E.D.Tymchatyn, On the dimension of certain totally disconnected spaces, Proc.A.M.S. 122(1994), 885-891. MR **95b:**54040
5. J.H.Roberts, The rational points in Hilbert space, Duke Math.Journ. 23 (1956), 489-492. MR **18:**55b

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