

THE INDEX NUMBER OF AN  $R$ -SPACE:  
AN EXTENSION OF A RESULT OF M. TAKEUCHI'S

CRISTIÁN U. SÁNCHEZ

(Communicated by Roe W. Goodman)

ABSTRACT. M. Takeuchi proved the following nice result: The “two-number” of a symmetric  $R$ -space is equal to the sum of the Betti numbers of the space with coefficients in  $Z_2$ . In the present paper an extension of this result is given for general  $R$ -spaces.

1. INTRODUCTION

In [Ch-N] a numerical invariant associated to compact Riemannian manifolds is introduced. This invariant, called the “2-number”, seems very difficult to compute except for compact symmetric spaces. Chen and Nagano obtained the 2-number for a great deal of them and at the same time showed the existence of many interesting connections between this number and topological invariants of the manifold. For a compact connected symmetric space  $M$  this number, denoted  $\#_2(M)$ , is defined as the maximal possible cardinality of the subsets  $A \subset M$  with the property that for each  $x \in A$  the symmetry  $s_x$  of  $M$  at  $x$  fixes every point of  $A$ . It is not hard to see that  $\#_2(M)$  is finite.

In [Ch-N, p. 274] the authors mention a beautiful result obtained by M. Takeuchi which appeared later in [T] and which is the main motivation of the present article. The result, where  $\beta_i(M, Z_2)$  denotes the  $i$ th Betti number of  $M$  with coefficients in  $Z_2$ , is the following.

**Theorem 1** (Takeuchi). *Let  $M$  be a symmetric  $R$ -space. Then*

$$\#_2(M) = \sum_{i \geq 0} \beta_i(M, Z_2). \quad \square$$

The difficulties of computing  $\#_2(M)$  for anything different from a symmetric space motivated the paper [S], where an extension of the invariant of Chen-Nagano was introduced for flag manifolds. As is well known, they have always a  $k$ -symmetric structure (see [K] and references therein) for some  $k \geq 2$  and one may define the “ $k$ -number”  $\#_k(M)$  of a complex flag manifold, [S, p. 1238], as the maximal possible cardinality of the subsets  $A_k \subset M$  with the property that for each  $x \in A_k$  the symmetry  $\theta_x$  fixes every point of  $A_k$ . If  $k = 2$ , i.e.,  $M$  is a Hermitian symmetric space, then clearly  $\#_k(M)$  is just the 2-number of  $M$ . Now corollary (3.6) of [S] can be stated as

---

Received by the editors August 5, 1994 and, in revised form, June 12, 1995.

1991 *Mathematics Subject Classification*. Primary 53C30; Secondary 53C35.

The author's research was partially supported by CONICET and CONICOR, Argentina.

**Theorem 2.** *If  $M$  is a complex flag manifold, then for any of its  $k$ -symmetric structures*

$$\#_k(M) = \sum_{i \geq 0} \beta_i(M, Z_2). \quad \square$$

Since complex flag manifolds are also  $R$ -spaces (see the next section), we see that there are two families of  $R$ -spaces, namely symmetric  $R$ -spaces and complex flag manifolds, where essentially the same result is true. Then one cannot help wondering whether there is a general theorem for  $R$ -spaces having Theorems 1 and 2 as particular cases. Naturally the main difficulty for this is that, in general,  $R$ -spaces have no “symmetries”.

The objective of this paper is to present a version of this general theorem (see Theorem 4 below). The paper is organized as follows. In section 2 we recall the definition of  $R$ -space (sometimes called orbit of an  $s$ -representation) and prove a fact which in some sense is implicit in certain definitions of  $R$ -space. Namely every  $R$ -space can be isometrically imbedded in a complex flag manifold (see Proposition 3). We have not seen this fact explicitly mentioned in the literature. This leads us to the natural definition of the *index of an  $R$ -space* and that of its *index number*. The main theorem is stated at the end of section 2 and proved in section 3. Section 4 contains an appendix.

## 2. BASIC FACTS AND MAIN RESULT

Let us recall the definition of an  $R$ -space. Let  $\mathcal{G}$  be a real semisimple Lie algebra without compact factors,  $\mathcal{K}$  a maximal compactly imbedded subalgebra of  $\mathcal{G}$  and  $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$  the Cartan decomposition of  $\mathcal{G}$  relative to  $\mathcal{K}$ . If we denote by  $B$  the Killing form of  $\mathcal{G}$ , then  $\mathcal{P}$  can be considered a Euclidean space with the inner product induced by the restriction of  $B$  to  $\mathcal{P}$ . Let  $G = \text{Int}(\mathcal{G})$  be the group of inner automorphisms of  $\mathcal{G}$ . The Lie algebra of  $G$  may be identified with  $\mathcal{G}$ . Let  $K$  be the analytic subgroup of  $G$  corresponding to  $\mathcal{K}$ ; then  $K$  is compact and acts on  $\mathcal{P}$  as an isometry group. By definition, an  $R$ -space  $M$  is the orbit of a non-zero vector  $E \in \mathcal{P}$  by  $K$ , i.e.,  $M = \text{Ad}(K)E$ . If  $E$  satisfies the relation  $(\text{ad}(E))^3 = \text{ad}(E)$  in  $\mathcal{G}$ , then the  $R$ -space  $M$  is symmetric and it is called a *symmetric  $R$ -space* (see for instance [F]). We consider in  $M$  the Riemannian metric induced from the inner product on  $\mathcal{P}$ .

$R$ -spaces may also be defined as homogeneous spaces of the form  $G/P$  where  $G$  is a real semisimple Lie group without compact factors and  $P$  is a parabolic subgroup. Here  $G/P \equiv K/K \cap P$  which is a  $K$ -orbit on  $\mathcal{P}$ . This seems to be a more standard definition; many authors call them *real flag manifolds*.

Every complex flag manifold may be considered an  $R$ -space. In fact let  $U$  be a compact connected semisimple centerless Lie group and  $\mathcal{U}$  its Lie algebra. The complex flag manifolds of  $U$  are the orbits of the adjoint action of  $U$  on  $\mathcal{U}$ . Take  $M = \text{Ad}(U)Y, Y \neq 0$  in  $\mathcal{U}$ . Let  $\mathcal{G} = \mathcal{U}_c = \mathcal{U} \oplus i\mathcal{U}$  then this is a Cartan decomposition of the realification  $\mathcal{G}_R$  of  $\mathcal{G}$  [H] and we may consider  $M$  as the orbit of  $iY$  in  $i\mathcal{U}$  by the adjoint action of  $U$ .

It is well known (see for instance [B], [W], [S1]) that the integral homology of complex flag manifolds is nonzero only in even dimensions and torsion free. For this reason their Betti numbers are the same for any choice of coefficient field.

**Proposition 3.** *Let  $M$  be an arbitrary  $R$ -space. Then there exists a complex flag manifold  $M_c$  such that  $M$  is isometrically imbedded in  $M_c$ . If  $M$  happens to be a symmetric  $R$ -space, then  $M_c$  is a Hermitian symmetric space and the isometric imbedding is totally geodesic. If  $M$  is already a complex flag manifold, then  $M_c = M$ .*

*Proof.* Associated to  $M$  we have the data indicated above, namely  $\mathcal{G}$ , semisimple without compact factors,  $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$  a Cartan decomposition,  $K \subset G = \text{Int}(\mathcal{G})$ ,  $B$  the Killing form and  $E \in \mathcal{P}$  such that  $M = \text{Ad}_G(K)E$ . Let  $a \subset \mathcal{P}$  be a maximal abelian subspace with  $E \in a$ . We may extend  $a$  to a Cartan subalgebra  $\mathcal{H} = t \oplus a$  of  $\mathcal{G}$  by taking as  $t$  a Cartan subalgebra of the centralizer of  $a$  in  $\mathcal{K}$ . Let  $\mathcal{G}_c$  be the complexification of  $\mathcal{G}$  and  $\sigma$  the conjugation of  $\mathcal{G}_c$  with respect to  $\mathcal{G}$ . Since  $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$  is a Cartan decomposition, there exists a compact real form  $\mathcal{G}_u$  of  $\mathcal{G}_c$  such that

$$\begin{aligned} \sigma\mathcal{G}_u &\subset \mathcal{G}_u, \\ \mathcal{K} &= \mathcal{G} \cap \mathcal{G}_u, \\ \mathcal{P} &= \mathcal{G} \cap i\mathcal{G}_u, \\ \mathcal{G}_u &= \mathcal{K} \oplus i\mathcal{P}. \end{aligned}$$

Let  $G_c$  be the simply connected semisimple Lie group associated to  $\mathcal{G}_c$ . Let  $G_1$  and  $G_u$  be the analytic subgroups of  $G_c$  corresponding to the subalgebras  $\mathcal{G}$  and  $\mathcal{G}_u$  respectively. By [H, p. 152, 4(ii)] they are both closed in  $G_c$  and by [H, p. 257]  $G_u$  is also simply connected. Let  $K_1$  be the analytic subgroup of  $G_c$  corresponding to  $\mathcal{K}$ . The group  $K_1$  is compact and clearly  $K_1 \subset G_1 \cap G_u$ . Now  $\text{Ad}_{G_1} : G_1 \rightarrow G$  is a surjective analytic homomorphism and  $K = \text{Ad}_{G_1}(K_1)$ . The manifold  $\text{Ad}_{G_1}(K_1)E$  is clearly  $M$ .

On the other hand, the orbit  $\text{Ad}_{G_1}(K_1)iE \subset i\mathcal{P}$  is again our space  $M$  since the representations of  $K_1$  on  $\mathcal{P}$  and  $i\mathcal{P}$  are equivalent. By [H, p. 180] if  $X, Y \in \mathcal{P}$

$$-B_u(iX, iY) = -B_c(iX, iY) = B_c(X, Y) = B(X, Y),$$

and so if we put on  $i\mathcal{P}$  the Euclidean metric induced by  $-B_u$ , then  $M$  and  $\text{Ad}_{G_1}(K_1)iE$  are isometric.

Let  $M_c$  be the orbit of  $iE$  by  $\text{Ad}_{G_u}(G_u)$ . It is obviously a complex flag manifold. Now consider on  $M_c$  the Riemannian metric induced by the inner product on  $\mathcal{G}_u$  defined by  $(-B_u)$ . It is clear that  $M$  is isometrically imbedded in  $M_c$ .

If  $M$  is a symmetric  $R$ -space, then, by definition, our original vector  $E \in \mathcal{P}$  satisfies  $\text{ad}(E)^3 = \text{ad}(E)$  in  $\mathcal{G}$ . Then in  $\mathcal{G}_u$  the vector  $iE$  satisfies  $\text{ad}(iE)^3 = -\text{ad}(iE)$  and it is not hard to see that under this condition, the orbit  $\text{Ad}_{G_u}(G_u)(iE)$  is a Hermitian symmetric space and  $T_{(iE)}(M) \subset T_{(iE)}(M_c)$  is a Lie triple system.

If  $M$  happens to be a complex flag manifold, then we may write it as  $M = U/L$  where  $U$  is a compact semisimple simply connected Lie group and  $L$  the centralizer of a torus in  $U$ . If  $\mathcal{U}$  denotes the Lie algebra of  $U$  we define  $\mathcal{G}_c = \mathcal{U} \oplus i\mathcal{U} = \mathcal{U}_c$ . If we consider the realification  $(\mathcal{G}_c)^R$  of  $\mathcal{G}_c$  we have that  $(\mathcal{G}_c)^R = \mathcal{U} \oplus i\mathcal{U} = \mathcal{K} \oplus \mathcal{P}$  is a Cartan decomposition [H, p. 185]. To construct the imbedding of  $M$  we take some element  $E \in z(\mathcal{L}) \subset \mathcal{L} = \text{Lie}(L)$ , such that  $\{\text{exp } tE : t \in R\}^- = Z(L)$  (the center of  $L$ ) and consider again the orbit of  $iE$  in  $i\mathcal{U}$ . Now by repeating the above construction in this case, we get the complexification of  $(\mathcal{G}_c)^R$ , namely  $[(\mathcal{G}_c)^R]_c = \mathcal{G}_c \oplus \mathcal{G}_c$  and its compact real form is just  $[(\mathcal{G}_c)^R]_u = \mathcal{U} \oplus \mathcal{U}$ . Now by keeping track of our original  $E \in z(\mathcal{L})$ ,  $iE \in i\mathcal{U}$  and  $i(iE) = -E \in \mathcal{U}$ , we see that

the complex flag manifold  $M_c$  is the adjoint orbit of  $-E = (0, -E) \in \mathcal{U} \oplus \mathcal{U}$  by  $U \times U$ , i.e.,  $M_c = \frac{U \times U}{U \times L} = U/L = M$  and therefore nothing new is obtained in this case. This completes the proof of Proposition 3.  $\square$

Let us consider now the flag manifold  $M_c$ . It is well known (see for instance [J, p. 135, A]) that there exists a natural number  $n_o = n_o(M_c) \geq 2$  such that for each  $n \geq n_o$ ,  $M_c$  is an  $n$ -symmetric manifold; see for instance [K]. This takes us naturally to the definition of the index of the  $R$ -space  $M$ . We define  $\text{Index}(M)$  as the first prime number  $p$  greater than or equal to  $n_o(M_c)$ .

Let  $\{\theta_x : x \in M_c\}$  be the  $p$ -symmetric structure on  $M_c$ . We are ready to define the index number of the  $R$ -space  $M$ , denoted by  $\#_I(M)$ , as follows.  $\#_I(M)$  is the maximal possible cardinality of the subsets  $A_p \cap M$  where  $A_p \subset M_c$  has the property that for each  $x \in A_p$  the symmetry  $\theta_x$  (of order  $p$  of  $M_c$ ) fixes every point of  $A_p$ .

We must point out that if  $M$  is a symmetric  $R$ -space, then by Proposition 3  $\text{Index}(M) = 2$  while if  $M$  is a complex flag manifold we have that its  $\text{Index}$  is the first prime number greater than or equal to  $n_o(M)$ . Obviously in the latter case,  $\#_I(M) = \#_p(M)$  as defined in [S, p. 1238]. In the former case, it is not obvious that  $\#_I(M) = \#_2(M)$ , but we shall see however that this is indeed true.

Now we may write down our mentioned extension of Theorem 1.

**Theorem 4.** *Let  $M$  be an  $R$ -space. Then*

$$\#_I(M) = \sum_{i \geq 0} \beta_i(M, Z_2). \quad \square$$

Notice that this shows in particular that for symmetric  $R$ -spaces we have  $\#_I(M) = \#_2(M)$ . The proof of the theorem is contained in the next section.

### 3. PROOF OF THE MAIN RESULT

Let  $M$  be our  $R$ -space,  $M_c$  its complex flag manifold and  $p = \text{Index}(M)$ . As was indicated in [S, p. 1238], it is easy to see that  $\#_p(M_c)$  is finite. Let  $A \subset M_c$  be a subset which satisfies the following property:

(P) *For each  $x \in A$  the symmetry  $\theta_x$  fixes every point of  $A$ .*

If the cardinality of  $A$  is precisely  $\#_p(M_c)$  we shall say that this subset is a  $\#_p$ -set of  $M_c$ .

**Lemma 5.** *Let  $A \subset M_c$  be a  $\#_p$ -set. Then there exists a maximal torus  $T_o \subset G_u$  such that  $A$  is the fixed point set  $F(T_o, M_c)$  of the torus  $T_o$  in  $M_c$ . Furthermore, if  $A_1 \subset M_c$  is a subset which satisfies the property (P), then there exists a  $\#_p$ -set  $A$  such that  $A_1 \subset A$ .*

*Proof.* Consider, for each  $a \in A$ , the isotropy subgroup  $(G_u)_a$  of  $G_u$  at  $a$  and denote  $Q = \bigcap_{a \in A} (G_u)_a$ . The group  $Q$  is connected and of maximal rank in  $G_u$  (see the Appendix). For any Lie group  $H$  let  $Z(H)$  denote its center.  $Z(Q)$  is contained in all the maximal tori of  $Q$ . Let  $T_o$  be a maximal torus in  $Q$ ; it is then a maximal torus in  $G_u$ . Let  $A_o = F(T_o, M_c)$ . It is well known that  $A_o$  is finite.

Notice now that  $A \subset A_o$ . In fact, for each  $a \in A$ , we know  $\theta_a \in Z[(G_u)_a] \subset T_o$  and therefore  $g\theta_a = \theta_a g, \forall g \in T_o$ , and since  $\theta_a$  is a symmetry and  $T_o$  is connected we must have  $g(a) = a, \forall g \in T_o$ .

On the other hand if  $x \in A_o$ , then  $\theta_x$  fixes every point of  $A_o$  because  $T_o \subset (G_u)_x$  and  $\theta_x \in Z[(G_u)_x] \subset T_o$ . So if  $A_o$  contains  $A$  properly, then  $A$  cannot be a  $\#_p$ -set. Then  $A = A_o$ .

Let  $A_1$  be a subset of  $M_c$  which satisfies (P) and consider again the subgroup  $Q_1 = \bigcap_{a \in A_1} (G_u)_a$ , connected and of maximal rank in  $G_u$ . If  $T_o$  is a maximal torus in  $Q_1$ , then  $A_1 \subset A_o = F(T_o, M_c)$ . This concludes the proof of the lemma.  $\square$

**Corollary 6.** *Let  $A \subset M_c$  be a  $\#_p$ -set. Then there exists a Cartan subalgebra  $\mathcal{H}_{u_o} \subset \mathcal{G}_u$  such that  $A = \mathcal{H}_{u_o} \cap M_c$ .*  $\square$

The converse of this corollary is clear: given any Cartan subalgebra  $\mathcal{H}_u$  in  $\mathcal{G}_u$ ,  $\mathcal{H}_u \cap M_c$  is a  $\#_p$ -set.

Let  $ia \subset i\mathcal{P}$  be a maximal abelian subspace. We may complete  $ia$  to a Cartan subalgebra  $\mathcal{H}_u = t \oplus ia \subset \mathcal{G}_u$  by taking as  $t$ , in the usual fashion, a Cartan subalgebra of the centralizer of  $ia$  in  $\mathcal{K}$  [M].

By extension of the name usually given in the non-compact case, we call this Cartan subalgebra of  $\mathcal{G}_u$  *minimally compact*.

Since the cardinality of the subsets  $A \subset M_c$  with the property (P) cannot be greater than  $\#_p(M_c)$ , we see that  $\#(A \cap M) \leq \#_p(M_c)$  for each set  $A$  with this property, where  $\#(X)$  indicates the cardinality of  $X$ . If  $A$  is a subset of  $M_c$  with the property (P) and such that  $\#(A \cap M)$  is maximal, then there is a  $\#_p$ -set  $A'$  in  $M_c$  such that

$$\#(A \cap M) = \#(A' \cap M),$$

then if  $\#(A \cap M)$  is maximal we may think that  $A$  is a  $\#_p$ -set in  $M_c$ .

**Lemma 7.** *Let  $A \subset M_c$  be a  $\#_p$ -set. Then there exists a minimally compact Cartan subalgebra  $\mathcal{H}_u \subset \mathcal{G}_u$  such that  $A' = \mathcal{H}_u \cap M_c$  satisfies  $A \cap M \subseteq A' \cap M$ .*

*Proof.* Let  $A \subset M_c$  be a  $\#_p$ -set. By Lemma 5 there exists a Cartan subalgebra  $\mathcal{H}_{u_o} \subset \mathcal{G}_u$  such that  $A = \mathcal{H}_{u_o} \cap M_c$ . Now

$$M \cap A = M \cap (M_c \cap \mathcal{H}_{u_o}) = M \cap \mathcal{H}_{u_o} = M \cap (\mathcal{H}_{u_o} \cap i\mathcal{P}).$$

Let  $ib = (\mathcal{H}_{u_o} \cap i\mathcal{P})$  (this may be  $\{0\}$ ). We may extend  $ib$  to a maximal abelian subspace  $ia \supseteq ib$  in  $i\mathcal{P}$  and construct with  $ia$  a minimally compact Cartan subalgebra  $\mathcal{H}_u$ . By construction  $M \cap A \subset M \cap \mathcal{H}_u$ . Then  $\mathcal{H}_u$  has the required property.  $\square$

**Lemma 8.** *If  $\mathcal{H}_u$  is minimally compact and  $A = \mathcal{H}_u \cap M_c$ , then  $\#(A \cap M)$  is maximal.*

*Proof.* Let us assume that there exists a minimally compact Cartan subalgebra  $\mathcal{H}_{u_o}$  such that  $A_o = \mathcal{H}_{u_o} \cap M_c$  and  $\#(A_o \cap M)$  is not maximal, i.e., there exists a subset  $A_1 \subset M_c$  which satisfies (P) and

$$(1) \quad \#(A_o \cap M) < \#(A_1 \cap M).$$

By Lemma 5 given  $A_1$  there exists a  $\#_p$ -set  $A_2$  in  $M_c$  such that  $A_1 \subset A_2$  in  $M_c$ . Then for  $A_2$  we have

$$\#(A_1 \cap M) < \#(A_2 \cap M).$$

By Lemma 7 there is a minimally compact Cartan subalgebra  $\mathcal{H}_{u_3}$  such that if we define  $A_3 = \mathcal{H}_{u_3} \cap M_c$ , then  $(A_2 \cap M) \subset (A_3 \cap M)$ , and then

$$\#(A_2 \cap M) \leq \#(A_3 \cap M).$$

Now write  $\mathcal{H}_{u_j} = t_j \oplus ia_j$  for  $j = 0, 3$  and consider

$$\begin{aligned} M \cap A_o &= M \cap (\mathcal{H}_{u_o} \cap i\mathcal{P}) = M \cap ia_o = B_o, \\ M \cap A_3 &= M \cap (\mathcal{H}_{u_3} \cap i\mathcal{P}) = M \cap ia_3 = B_3. \end{aligned}$$

Since  $ia_o$  and  $ia_3$  are conjugate by an element  $q \in K$  by [H, Ch. V: 6, 6.3],  $\#(B_o) = \#(B_3)$ . This contradicts inequality (1) and the lemma follows.  $\square$

Now let  $\mathcal{H}_u$  be a fixed minimally compact Cartan subalgebra of  $\mathcal{G}_u$  containing  $iE$  and let  $N_{G_u}(\mathcal{H}_u)$  denote the normalizer of  $\mathcal{H}_u$  in the compact simply connected Lie group  $G_u$ . As in Proposition 3 denote by  $\sigma$  the conjugation of  $\mathcal{G}_c$  with respect to  $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$  and by  $\tau$  the conjugation with respect to  $\mathcal{G}_u = \mathcal{K} \oplus i\mathcal{P}$ . Notice that  $\mathcal{H}_u$  is  $\sigma$ -invariant. Let us denote by  $[N_{G_u}(\mathcal{H}_u)]_\sigma$  the subgroup of those elements which commute with  $\sigma$  on  $\mathcal{H}_u$ . This is the same as the subgroup of those elements commuting with  $\theta = \sigma\tau$  since  $\tau$  is the identity on  $\mathcal{G}_u$ . Clearly the orbit  $N_{G_u}(\mathcal{H}_u)(iE)$  is  $M_c \cap \mathcal{H}_u$  and hence a  $\#_p$ -set in  $M_c$ . Let  $N_{K_1}(ia)$  denote the normalizer of  $ia$  in the group  $K_1$ .

**Lemma 9.**  $N_{G_u}(\mathcal{H}_u)(iE) \cap M = [N_{G_u}(\mathcal{H}_u)]_\sigma(iE) = N_{K_1}(ia)(iE)$ .

*Proof.* First we show that  $N_{G_u}(\mathcal{H}_u)(iE) \cap M \subset [N_{G_u}(\mathcal{H}_u)]_\sigma(iE)$ . Take  $g \in N_{G_u}(\mathcal{H}_u)$  such that  $\text{Ad}(g)(iE) \in M$ . Then there exists  $h \in K_1$  such that  $\text{Ad}(h)(iE) = \text{Ad}(g)(iE)$ . Since  $\text{Ad}(g)(iE) \in \mathcal{H}_u$  and  $M \subset i\mathcal{P}$ , we have  $\text{Ad}(g)(iE) \in \mathcal{H}_u \cap i\mathcal{P} = ia$  and hence  $\text{Ad}(h)(iE) \in ia$ . By [H, p. 285, (2.2)] there exists  $s \in W(G_u, K_1)$  (notation [H, p. 284]) such that  $s(iE) = \text{Ad}(h)(iE)$ . Now there exists, by [H, p. 325, (8.10)],  $g_1 \in [N_{G_u}(\mathcal{H}_u)]_\sigma$  such that  $\text{Ad}(g_1)(iE) = s(iE)$  and this proves the indicated inclusion. To prove the other one we notice that it is enough to show that  $[N_{G_u}(\mathcal{H}_u)]_\sigma(iE) \subset M$  and this is clearly a by-product of the second equality. Then to finish our proof we just have to show  $[N_{G_u}(\mathcal{H}_u)]_\sigma(iE) = N_{K_1}(ia)(iE)$ .

Clearly  $[N_{G_u}(\mathcal{H}_u)]_\sigma(iE) \subset N_{G_u}(ia)(iE)$  (the normalizer of  $ia$  in  $G_u$ ). Now take  $g \in N_{G_u}(ia)$ , by [H, p. 324, (8.8)] we have  $g = uv$  where  $u \in N_{K_1}(ia)$  and  $v \in A_*$  (notation in [H, p. 324]).

*Remark.* Notice that  $G_u$  is simply connected and therefore [H, p. 324, (8.8)] can be applied.

Then  $\text{Ad}(g)(iE) = \text{Ad}(u)(iE)$  and this proves

$$[N_{G_u}(\mathcal{H}_u)]_\sigma(iE) \subset N_{G_u}(ia)(iE) \subset N_{K_1}(ia)(iE).$$

Now to prove the other inclusion take  $h \in N_{K_1}(ia)$ . We have  $\text{Ad}(h)t = t'$  and in general  $t \neq t'$ . But  $t$  and  $t'$  are both Cartan subalgebras of  $M = \text{Lie}(N_{K_1}(ia)) = \text{Lie}(Z_{K_1}(ia))$  (these are  $M$  and  $M'$  in the notation of [H, p. 284]) and therefore by the Theorem of conjugation [H, p. 248, (6.4) (ii)] there exists  $w \in [Z_{K_1}(ia)]_e$ , the connected component of  $e$  in  $[Z_{K_1}(ia)]$ , such that  $\text{Ad}(w)t' = t$ . Let us now define  $g = wh$ ; it is clear that  $g$  normalizes both  $t$  and  $ia$ . Hence  $g$  normalizes  $\mathcal{H}_u$  and commutes with  $\sigma$  there and therefore  $g \in [N_{G_u}(\mathcal{H}_u)]_\sigma$ .

Since  $\text{Ad}(g)(iE) = \text{Ad}(h)(iE)$ , we have  $N_{K_1}(ia)(iE) \subset [N_{G_u}(\mathcal{H}_u)]_\sigma(iE)$  and this completes the proof of the lemma.  $\square$

It is clear now that Lemma 9 together with [T-K, p. 205, (2.1)] yield the identity

$$\#((N_{G_u}(\mathcal{H}_u)(iE)) \cap M) = \sum_{i \geq 0} \beta_i(M, Z_2).$$

Theorem 4 now follows from this equality and Lemma 8.  $\square$

## 4. APPENDIX

Here we give a proof of the following fact used in Lemma 5.

**Lemma 10.** *The group  $Q$  (defined in the proof of Lemma 5) is connected and has maximal rank in  $G_u$ .*

*Proof.* Let  $A = \{a_1, \dots, a_r\}$ ,  $r \geq 1$ , be the  $\#_p$ -set in  $M_c$  considered in Lemma 5. To simplify notation we write  $K_j = (G_u)_{a_j}$  and let  $\mathcal{K}_j$  denote their respective Lie subalgebras in  $\mathcal{G}_u$ . The subgroups  $K_j$  (which are conjugate to each other) are centralizers of tori in  $G_u$ .

For each  $j = 1, \dots, r$ , let us decompose  $\mathcal{G}_u = \mathcal{K}_j \oplus \mathcal{M}_j$  orthogonally with respect to  $(-B_u)$  and consider the following observation.

(\*)  $\text{For each } j = 1, \dots, r, A \subset \mathcal{K}_j.$

In fact this is obvious for  $a_j$  but it is also true for  $a_t$  for  $t \neq j$  because we may write  $a_t = v_t + h_t$  with  $v_t \in \mathcal{K}_j$  and  $h_t \in \mathcal{M}_j$ . Since  $h_t \in \mathcal{M}_j = T_{a_j}(M_c)$  and  $\theta_{a_j}(a_t) = a_t$  ( $A$  satisfies property (P)) we must have  $h_t = 0$  because  $\theta_{a_j}$  is a symmetry at  $a_j$ . Then  $a_t \in \mathcal{K}_j$ .

Let us now take  $a_2 \in \mathcal{K}_1$ . Its orbit by the adjoint action of  $K_1$  on its Lie algebra  $\mathcal{K}_1$  is  $K_1(a_2) = K_1/(K_1 \cap K_2)$  and since this is a complex flag manifold in  $\mathcal{K}_1$  we have that  $(K_1 \cap K_2)$  is connected and has maximal rank in  $K_1$  (hence in  $G_u$ ).

To proceed by induction let us assume that we have shown that the subgroup  $K_1 \cap K_2 \cap \dots \cap K_j$  is connected and of maximal rank in  $G_u$  and consider  $a_{j+1}$  in  $A$ . By (\*) we have  $a_{j+1} \in (K_1 \cap K_2 \cap \dots \cap K_j)$  and hence its orbit by the adjoint action of the group  $(K_1 \cap K_2 \cap \dots \cap K_j)$  in the Lie algebra  $(\mathcal{K}_1 \cap \mathcal{K}_2 \cap \dots \cap \mathcal{K}_j)$  is a complex flag manifold. Namely

$$(K_1 \cap K_2 \cap \dots \cap K_j)(a_{j+1}) = \frac{(K_1 \cap K_2 \cap \dots \cap K_j)}{(K_1 \cap K_2 \cap \dots \cap K_j \cap K_{j+1})},$$

and hence  $K_1 \cap K_2 \cap \dots \cap K_{j+1}$  is connected and of maximal rank in  $K_1 \cap K_2 \cap \dots \cap K_j$  and therefore in  $G_u$ . Then  $Q$  is connected and of maximal rank in  $G_u$ .  $\square$

In the proof of Lemma 10 we did not use the fact that  $A$  is a  $\#_p$ -set, only that it satisfies property (P).

## REFERENCES

- [B] R. Bott, *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France **84** (1956), 251–281. MR **19**:291a
- [Ch-N] B. Y. Chen and T. Nagano, *A Riemannian geometric invariant and its applications to a problem of Borel and Serre*, Trans. Amer. Math. Soc. **308** (1988), 273–297. MR **89j**:53046
- [F] D. Ferus, *Symmetric submanifolds of Euclidean space*, Math. Ann. **247** (1980), 81–93. MR **81i**:53040
- [H] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1978. MR **80k**:53081
- [J] J. A. Jimenez, *Existence of Hermitian  $n$ -symmetric spaces and of noncommutative naturally reductive spaces*, Math. Z. **196** (1987), 133–139. MR **89b**:53088a
- [K] O. Kowalski, *Generalized symmetric spaces*, Lecture Notes in Math., vol. 805, Springer, Berlin, Heidelberg, and New York, 1980. MR **83d**:53036
- [M] I. G. Macdonald, *Algebraic structure of Lie groups*, Representation Theory of Lie Groups, London Math. Soc. Lecture Note Ser., vol. 34, Cambridge University Press, 1979, pp. 91–150. MR **81j**:22001
- [S] C. U. Sanchez, *The invariant of Chen-Nagano on flag manifolds*, Proc. Amer. Math. Soc. **118** (1993), 1237–1242. MR **93j**:53066

- [S1] C. U. Sanchez, *The tightness of certain almost complex submanifolds*, Proc. Amer. Math. Soc. **110** (1990), 807–811. MR **91b**:53070
- [T] M. Takeuchi, *Two-number of symmetric R-spaces*, Nagoya Math J. **115** (1989), 43–46. MR **90i**:53066
- [T-K] M. Takeuchi and S. Kobayashi, *Minimal imbeddings of R-spaces*, J. Differential Geometry **2** (1968), 203–215. MR **39**:366
- [W] J. Wolf, *The action of a real semisimple group on a complex flag manifold I: Orbit structure and holomorphic arc components*, Bull. Amer. Math. Soc. **75** (1969), 1121–1237. MR **40**:4477

FA. M.A.F., UNIVERSIDAD DE CÓRDOBA, CIUDAD UNIVERSITARIA, 5000, CÓRDOBA, ARGENTINA  
E-mail address: `csanchez@mate.uncor.edu`