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HILBERT C*−MODULES IN WHICH ALL CLOSED SUBMODULES ARE COMPLEMENTED

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Abstract. Let B be a C*−algebra. If there exists a full Hilbert B−module X such that $X = Y \oplus Y^\perp$ for each closed submodule Y, then B is *-isomorphic to a C*−algebra of (not neccesarily all) compact operators on a Hilbert space.

1. Introduction and the main result

It is well known that each closed subspace $\mathcal{K}$ of a Hilbert space $\mathcal{H}$ has an orthogonal complement $\mathcal{K}^\perp$ such that $\mathcal{K}\oplus\mathcal{K}^\perp = \mathcal{H}$. (In fact, by [7] a similar property characterizes Hilbert spaces among all Banach spaces.) This property, however, does not hold in general for closed submodules of a Hilbert module over a C*−algebra (see [5] or [9]; in [6] Hilbert modules that are complemented in any larger Hilbert module are characterized). In this note we shall characterize all C*−algebras B such that all closed submodules in any Hilbert module over B are orthogonally complemented. Here a submodule Y of a Hilbert module X is called orthogonally complemented if there exists a self-adjoint projection p on X such that $Y = pX$ (see [5, p. 21]). More precisely, we shall prove the following result.

Theorem 1. *Let B be a C*−algebra. If there exists a full Hilbert B−module in which every closed submodule is orthogonally complemented, then B is *-isomorphic to a C*−algebra of (not necessarily all) compact operators on some Hilbert space. Consequently, all closed submodules in all Hilbert B−modules are orthogonally complemented.*

2. Preliminaries

A *(right) Hilbert module* over a C*−algebra B is a right B−module X equipped with a B-valued inner product $\langle\ ,\ \rangle$, which is linear over B in the second and conjugate linear in the first variable, such that X is a Banach space with the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. Hilbert modules are important for operator K-theory, and the basic theory of such modules can be found in [4] and [5]. We denote by L (or by $L(X)$ if X is not clear from the context) the C*−algebra of all operators (necessarily bounded and linear over B) on X that have an adjoint. For arbitrary $x, y \in X$ there is an operator $[y, x] \in L$ defined by $[y, x]z = y\langle x, z\rangle$ $(z \in X)$. The closed linear span of such operators (which is the analogy for compact operators between

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Hilbert spaces) will be denoted by K (or $K(X)$). K is a closed two-sided ideal (not necessarily proper) in L.

For a general Hilbert $B-$module X the closed linear span of the set $\{\langle x,y\rangle : x,y \in X\}$ is a closed two-sided ideal in B and will be denoted by $\langle X,X\rangle$. If $\langle X,X\rangle = B$ then X is called a *full Hilbert $B-$module*.

The subset Λ of $L(X \oplus B)$ consisting of all matrices of the form

$$\begin{bmatrix} a & x \\ y^* & b \end{bmatrix},$$

where $a \in K,\ b \in B$ and $x,y \in X$, is a C$^*-$subalgebra. Here each $y \in X$ is regarded as an operator from B to X defined by $b \mapsto yb$, with the adjoint given by $y^*(x) = \langle y,x\rangle$. This algebra was introduced in [3] and called the *linking algebra of* X.

We shall regard B, X and $K = K$ as subsets of Λ in the obvious way:

$$B \cong \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}, \quad X \cong \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \text{ and } K \cong \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that the inner product $\langle x,y\rangle$ in X can be expressed as the product x^*y in Λ.

3. Proof of the theorem

To prove the theorem we shall relate submodules of X to right ideals of the linking algebra Λ of X, but first we need a couple of simple lemmas.

Lemma 1. *If X is a Hilbert $B-$module such that every closed submodule of X is orthogonally complemented, then each closed right ideal J of $K = K(X)$ is of the form $J = pK$ for some projection $p \in L = L(X)$.*

Proof. Consider the closed submodule $[JX]$ of X. By hypothesis $[JX] = pX$ for some projection $p \in L$. Then $pax = ax$ for all $x \in X$ and $a \in J$, hence $pa = a$ and consequently $J \subseteq pK$. To prove the reverse inclusion, it suffices to show that $p[x,y] = [px,y]$ is in J for each 'rank one' operator $[x,y] \in K$ (since K is spanned by such operators). Since px can be approximated by finite sums of elements of the form az $(a \in J,\ z \in X)$, $[px,y]$ can be approximated by finite sums of operators of the form $[az,y] = a[z,y]$, which are in J. This proves that $J = pK$. □

Lemma 2. *If every closed right ideal J in a C$^*-$algebra A is of the form $J = pA$ for some projection p in the multiplier algebra $M(A)$ of A, then A is *-isomorphic to a C$^*-$algebra of (not necessarily all) compact operators on some Hilbert space.*

Proof. First suppose that A is commutative, hence $A \cong C_0(T)$ and $M(A) \cong C_b(T)$ (= bounded continuous functions on a locally compact Hausdorff space T). By hypothesis for each $t \in T$ there exists a projection $p \in C_b(T)$ such that

$$\{f \in C_0(T) :\ f(t) = 0\} = pC_0(T).$$

Clearly such a p must be the characteristic function of the complement of $\{t\}$, hence each singleton in T is open and T is discrete. Consequently A is *-isomorphic to a C$^*-$algebra of (diagonal) compact operators on some Hilbert space.

In general, we shall prove that for each maximal commutative C^*- subalgebra A_0 of A every maximal ideal J_0 of A_0 is of the form $J_0 = pA_0$ for some projection $p \in M(A_0)$. By previous paragraph this implies that the maximal ideal space of A_0 is discrete, hence by [2, p. 84] A is *-isomorphic to a subalgebra of $K(\mathcal{H})$ for some

Hilbert space $\mathcal{H}$. Given a maximal ideal J_0 of A_0, consider the closed right ideal $J = [J_0 A]$ of A. By hypothesis $J = pA$ for some projection $p \in M(A)$. We may assume that $A \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ such that $[A\mathcal{H}] = \mathcal{H}$; then $M(A) = \{b \in A'' : \ bA + Ab \subseteq A\}$. Note that $[J_0\mathcal{H}] = [J_0 A\mathcal{H}] = [J\mathcal{H}] = [pA\mathcal{H}] = p\mathcal{H}$; in particular the range of p is invariant under A_0', therefore $p \in A_0''$. From $p \in M(A)$ it then follows that $pA_0 \subseteq A_0'' \cap A$. Since $A_0'' \cap A$ is a commutative C*−subalgebra of A and contains A_0, while A_0 is maximal commutative in A, $A_0'' \cap A = A_0$. Thus $pA_0 \subseteq A_0$, which implies that $p \in M(A_0)$.

Since the range of p is $[J_0\mathcal{H}]$, we have $J_0 \subseteq pA_0$; but, since J_0 is a maximal ideal in A_0, the required equality $J_0 = pA_0$ follows if $pA_0 \neq A_0$. To see that $pA_0 \neq A_0$, let ρ_0 be a character on A_0 with $\ker \rho_0 = J_0$ and let ρ be a pure state on A extending ρ_0. Then J is contained in the right kernel of ρ, hence A_0 is not contained in J. On the other hand, $pA_0 \subseteq J$ (since $J = pA$). □

Proof of Theorem 1. Let X be a full Hilbert module over B such that all closed submodules of X are orthogonaly complemented. Then by Lemmas 1 and 2 there exists a *-monomorphism $\pi_K : K \to K(\mathcal{H}_K)$ for some Hilbert space $\mathcal{H}_K$ (such that $[\pi_K(K)\mathcal{H}_K] = \mathcal{H}_K$). By a well known result concerning extensions of representations (see [8, p. 162]) there exist a Hilbert space $\mathcal{H} \supseteq \mathcal{H}_K$ and a representation π of the linking algebra Λ of X on $\mathcal{H}$ such that $\pi(a)|\mathcal{H}_K = \pi_K(a)$ for each $a \in K$. Since π is a $*$−representation we have

$$\pi(x)^*\pi(x) = \pi(\langle x, x\rangle) \quad \text{and} \quad \pi(x)\pi(x)^* = \pi([x, x]) \tag{1}$$

for all $x \in X$. Put $\mathcal{H}_B = [\pi(X)^*\mathcal{H}_K]$ and note that $\mathcal{H}_B$ is invariant under $\pi(B)$ since $BX^* = (XB)^* \subseteq X^*$. Since $BK = 0$, $KB = 0$, $XK = 0$ and $X^*X^* = 0$ (the products are computed in Λ), the subspaces $\mathcal{H}_K = [\pi(K)\mathcal{H}_K]$ and $\mathcal{H}_B$ are orthogonal and we have

$$\pi(B)\mathcal{H}_K = 0,\ \pi(K)\mathcal{H}_B = 0,\ \pi(X)\mathcal{H}_K = 0 \text{ and } \pi(X)^*\mathcal{H}_B = 0. \tag{2}$$

Moreover,

$$[\pi(X)\mathcal{H}_B] = [\pi(X)\pi(X)^*\mathcal{H}_K] = [\pi(K)\mathcal{H}_K] = \mathcal{H}_K.$$

These relations easily imply that $\mathcal{H}_K \oplus \mathcal{H}_B$ is invariant under π; hence, replacing π by the corresponding restriction, we may assume that $\mathcal{H}_K \oplus \mathcal{H}_B = \mathcal{H}$. Then the second identity in (1) (together with the second identity in (2) and the fact that $\pi_K(K)$ consists of compact operators) implies that $\pi(x)$ is compact for all $x \in X$. Since $B = \langle X, X\rangle$, it follows then from the first identity in (1) that $\pi(B)$ consists of compact operators only. Moreover, since π_K is one to one, it is easy to see (using (1)) that the restriction of π to (X and to) B is one to one. (If $J \stackrel{\text{def}}{=} \ker \pi|B \neq 0$, then $[XJ, XJ]$ would be a non-zero ideal of K contained in $\ker \pi_K$.) Thus B is *-isomorphic to a C*−subalgebra of $K(\mathcal{H}_B)$.

Conversely, suppose that B is isomorphic to a (nondegenerate) C*−algebra of compact operators on a Hilbert space $\mathcal{H}_B$ by a *-isomorphism π_B. Let π be a representation of Λ on a Hilbert space $\mathcal{H} \supseteq \mathcal{H}_B$ such that $\pi(b)|\mathcal{H}_B = \pi_B(b)$ for each $b \in B$, and put $\mathcal{H}_K = [\pi(X)\mathcal{H}_B]$. As in the previous paragraph, we may assume that $\mathcal{H}_K \oplus \mathcal{H}_B = \mathcal{H}$. Since π_B is one to one it follows (using (1)) that the restrictions of π to X and K are one to one, but then we see (using (2)) that π is also one to one. Thus, Λ is *-isomorphic to a C*−subalgebra of $K(\mathcal{H})$, hence by [1, 1.4.5] Λ is *-isomorphic to a (restricted) direct sum of elementary C*−algebras of the form $K(\mathcal{H}_j)$ for some Hilbert spaces $\mathcal{H}_j$. Since $K(\mathcal{H}_j)$ is strongly Morita equivalent to $\mathbb{C}$, it would not be hard to see (using induced representations, see [10] or [5]) that $K(\mathcal{H}_j)$ and $\mathbb{C}$ have equivalent categories of Hilbert modules, from which it is possible to conclude the proof, but we shall give here a more elementary approach. Since Λ is *-isomorphic to a direct sum of algebras of the form $K(\mathcal{H}_j)$, each closed right ideal J of Λ is of the form $J = p_\Lambda \Lambda$ for some projection p_Λ in the multiplier algebra $M(\Lambda)$ of Λ. Using the above representation of Λ and the well known fact that the multiplier algebra of K is L (see [5]), it is easy to see that $M(\Lambda)$ is of the form

$$M(\Lambda) = \begin{bmatrix} L & Z \\ Z^* & M(B) \end{bmatrix},$$

where Z is a certain $L - M(B)-$subbimodule of $B(\mathcal{H}_B, \mathcal{H}_K)$ containing X. Given a closed submodule Y of X, consider the closed right ideal J of Λ generated by Y, that is

$$J = \begin{bmatrix} [Y, X] & Y \\ 0 & 0 \end{bmatrix}.$$

If p_Λ is a projection in $M(\Lambda)$ such that $J = p_\Lambda \Lambda$, then a straightforward computation shows that p_Λ is necessarily of the form

$$p_\Lambda = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$$

for some projection $p \in L$, and that $Y = pX$. $\square$

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