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GROUP ALGEBRAS WHOSE UNITS SATISFY A GROUP IDENTITY

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Abstract. Let FG be the group algebra of a torsion group over an infinite field F. Let U be the group of units of FG. We prove that if U satisfies a group identity, then FG satisfies a polynomial identity. This confirms a conjecture of Brian Hartley.

1. Introduction

The unit group $U = U(FG)$ is said to satisfy a group identity if there exists a nontrivial word $w(x_1, \ldots, x_m)$, in the free group generated by $x_1, \ldots, x_m$, such that $w(u_1, \ldots, u_m) = 1$ for all $u_i \in U$. Hartley suggested the following

Conjecture 1.1. *If G is a torsion group and $U(FG)$ satisfies a group identity, then FG satisfies a polynomial identity.*

We recall that FG is said to satisfy a polynomial identity (PI) if there exists a nonzero polynomial $f(y_1, \ldots, y_n) \in F\{y_1, \ldots, y_n\}$, in noncommuting variables such that $f(\alpha_1, \ldots, \alpha_n) = 0$ for all $\alpha_i \in FG$. Group algebras satisfying a PI were classified by Passman and Isaacs-Passman.

This conjecture was first studied by Warhurst [10] who investigated special words satisfied by $U(FG)$. Pere Menal [5] suggested a possible solution for some p-groups. We are able to use his construction. Giambruno-Jespers-Valenti [4] settled the case where G has no p-element if p is the characteristic of F. Further Goncalves-Mandel [3] and Dokuchaev-Goncalves [1] studied group algebras when U satisfies a semigroup identity. There have been many papers classifying groups so that U satisfied a special group identity like $(u_1, \ldots, u_n)$ or (u^n, v) and many others. The main result of this paper is the following theorem.

Theorem. *Suppose that F is an infinite field and that G is a torsion group. If $U(FG)$ satisfies a group identity, then FG satisfies a polynomial identity.*

2. Some lemmas

We write $(u, v) = u^{-1}v^{-1}uv$ for the multiplicative commutator and $[x, y] = xy - yx$ for the additive commutator. We denote by $\Delta(G, N)$ the kernel of $FG \to$

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$F(G/N)$ and $\Delta(G,G)=\Delta(G)$ is the augmentation ideal. Furthermore, let

$$P=\{g\in G\colon o(g)=p^k \text{ for some } k\}, \text{the } p\text{-elements},$$
$$Q=\{g\in G\colon p\nmid o(g)\},\ \text{ the } p'\text{-elements},$$

where $p=\operatorname{char} F$. If $p=0$, we let $P=\{1\}$, $G=Q$. We set

$$\phi=\phi(G)=\{g\in G\colon g \text{ has a finite number of conjugates in } G\}$$

to be the FC-subgroup of G, whereas

$$\phi_p=\phi_p(G)=\langle P\cap\phi\rangle.$$

We first record a couple of useful results and then observe what it means for a finite group G that $U(FG)$ satisfies a group identity.

Lemma 2.1. *Let R be a semiprime ring and let $S=\{a\in R : \text{for all } b,c\in R,\ bc=0 \text{ implies } bac=0\}$. If S contains all square-zero elements of R, then S contains all nilpotent elements of R.*

Proof. We shall prove that S contains all nilpotent elements of R by induction on the index of nilpotence. Let $a\in R$ be such that $a^n=0$, $a^{n-1}\neq 0$ and suppose by induction that $x\in R$, $x^m=0$ with $m<n$ implies $x\in S$. Let $b,c\in R$ be such that $bc=0$ and let $r\in R$. Then, since $b(1-a)^{-1}(1-a)c=0$ and $(crb)^2=0$, we get by hypothesis that $b(1-a)^{-1}crb(1-a)c=0$. Thus

$$b(1+a+a^2+\cdots+a^{n-1})crb(1-a)c=0.$$

Now, by the inductive hypotheses $a^2,a^3,\ldots,a^{n-1}\in S$ and we get

$$b(1+a+\cdots+a^{n-1})c=bac \quad\text{and}\quad b(1-a)c=-bac.$$

Thus $bacrbac=0$ for all $r\in R$. Since R is semiprime, then $bac=0$ and $a\in S$. $\square$

Corollary 2.2. *Let R be a semiprime algebra over an infinite commutative domain C. If $U(R)$ satisfies a group identity, then every idempotent of $C^{-1}R$ is central and for all $b,c\in R$ such that $bc=0$ we have $bac=0$ whenever a is any nilpotent element of R.*

Proof. Lemma 2.1 and Lemma 2 of [4]. $\square$

Recall that a group is called p-abelian if its derived group is a finite p-group.

Lemma 2.3. *Let G be a finite nonabelian group. Then U satisfies a group identity if and only if G is p-abelian and* char $F=p$*; in this case, U satisfies the identity $(u,v)^{p^n}=1$ for some n and FG satisfies the polynomial identity $[x,y]^{p^m}=0$ for some m.*

Proof. Let J be the Jacobson radical of FG. Then J is nilpotent and we have an epimorphism $FG\to FG/J$ which induces a homomorphism

$$U(FG)\to U(FG/J)=\overline{U}.$$

The latter map is onto as $uv=1-j$, $j\in J$, implies that $j^k=0$ and

$$uv(1+j+j^2+\cdots+j^{k-1})=1.$$

Since U satisfies a group identity, so does $\overline{U}$. Now FG/J, being semisimple, is a direct sum of matrix rings over division rings:

$$FG/J=\sum^{\oplus}(D)_{n\times n}.$$

Since FG/J is semiprime, by 2.2 all idempotents are central; hence in the decomposition above, n cannot be bigger than 1.

Moreover, D must be commutative because otherwise it contains a noncyclic free group as observed by Goncalves [2]. To see this one applies the

Tits' alternative. Any linear group is solvable by locally finite or it contains a noncyclic free group.

We have, therefore, seen that FG/J is commutative. It follows that $(x, y) = 1 \bmod J$ for all $x, y \in G$ and consequently $G' \subseteq 1 + J$. Thus $\Delta(G') \subseteq J$ is nilpotent and G' is a p-group with $p = \text{char } F$ as asserted.

Now, suppose that G' is a (finite) p-group so that $(\Delta G')^{p^k} = 0$ for some k. Then for all $u, v \in FG$ we have $(u, v) = 1 \mod \Delta(G, G')$ and $(u, v)^{p^k} \in 1+(\Delta(G, G'))^{p^k} = 1 + FG(\Delta G')^{p^k} = 1$. Moreover, for $x, y \in FG$

$$\begin{aligned}[x, y] &= \sum c[g, h] \qquad (c \in F,\ g, h \in G)\\ &= \sum cgh(1 - (g, h)) \in \Delta(G, G').\end{aligned}$$

It follows that $[x, y]^{p^k} = 0$. This completes the proof of the lemma. □

We shall need the following classical results.

Lemma 2.4. (a) (Isaacs-Passman) *If* char $K = 0$, *then KG satisfies a PI if and only if G contains an abelian subgroup of finite index.*

(b) (Passman) *If* char $K = p > 0$, *then KG satisfies a PI if and only if G contains a p-abelian subgroup of finite index.*

Proof. See [6, p. 196]. □

Lemma 2.5. (Passman) *Suppose that* char $K = p > 0$. *Then KG is semiprime if and only if $\phi(G)$ is a p'-group.*

Proof. See [6, p. 131]. □

Let $N = N(KG)$ be the sum of all nilpotent ideals of KG.

Lemma 2.6. (Passman) *Suppose that* char $K = p > 0$. *Then $N(KG)$ is nilpotent if and only if $\phi_p(G)$ is finite.*

Proof. See [6, p. 311]. □

3. Proof of the theorem

We shall divide the proof into the following three exhaustive and mutually exclusive cases

(i) FG is semiprime, i.e. $N = 0$.
(ii) N is (nonzero) nilpotent.
(iii) N is nil but not nilpotent.

Case (i): FG semiprime.

Let $y \in Q$ have order $m, p \nmid m$. Write

$$\hat{y} = 1 + y + \cdots + y^{m-1}.$$

Then $\hat{y}/m = e = e^2$ is central by 2.2. It follows that $\langle y \rangle$ is normal in G. Thus Q is abelian or Hamiltonian, namely, $K_8 \times A$ where K_8 is the quaternion group of order

8 and A is abelian. But K_8 cannot arise as FK_8 will have a summand which is a division ring or a ring of matrices, both not allowed as seen in 2.3. In any case, Q is abelian. If char $F = 0$, then $Q = G$ is abelian. Thus we may suppose that $p > 0$.

Let $g, h \in P$. Then $(1 - g)$ is nilpotent and $\hat{h}(1 - h) = 0$. Therefore, by 2.2, we have

$$0 = \hat{h}(1 - g)(1 - h) = \hat{h}g(1 - h).$$

It follows that $\hat{h}gh = \hat{h}g$ and, therefore, $g = h^i gh$ for some i. Thus $ghg^{-1} = h^{-i}$ and $\langle h \rangle \lhd P$. We have proved that P is a group which is abelian or Hamiltonian. We shall see, in a moment, that P will turn out to be trivial.

Take $\pi \in P$ and let $g \in Q$. Since $\langle g \rangle \lhd G$, it follows that $H = \langle g, \pi \rangle$ is a finite subgroup of G. By Lemma 2.3, H is p-abelian, so the commutator (g, π) is a p-element in $\langle g \rangle$. Hence $(g, \pi) = 1$ and it follows that $\langle \pi \rangle$ is normalized by Q and also by P. Thus $\langle \pi \rangle \lhd G$ and, since FG is semiprime, we have $\pi = 1$ and $G = Q$ is abelian.

Case (ii): N (nonzero) nilpotent.

Since N is nilpotent, $\phi_p(G)$ is a finite group by 2.6 and $p =$ char F. Then $\phi(G/\phi_p(G))$ has no p-elements. So $F(G/\phi_p(G))$ is semiprime.

To the finite group $\phi_p(G)$ we apply 2.3 to conclude that its commutator group is a p-group. It follows that the p-elements of $\phi_p(G)$ form a group. Since $\phi_p(G)$ is generated by p-elements it is a p-group.

Thus we have an epimorphism $U(FG) \to UF(G/\phi_p) = \overline{U}$. Since $\overline{U}$ satisfies a group identity and $F(G/\phi_p)$ is semiprime we can apply case (i) to conclude that G/ϕ_p is abelian. Thus G' is contained in ϕ_p and is a finite p-group. We have proved that FG satisfies a polynomial identity (of the form $[x, y]^{p^m} = 0$).

Case (iii): N nil but not nilpotent.

Let $R = F\{X\}$ be the free algebra on a countable set $X = \{x_1, x_2, \dots\}$. Let t be an indeterminate and $R[[t]]$ the power series ring over R. Finally, let $w = w(y_1, \dots, y_m)$ be the nontrivial group identity satisfied by $U(FG)$. Then the elements $1 + x_1 t, 1 + x_2 t, \dots, 1 + x_n t$ are units in $R[[t]]$. They generate a free group by the Magnus argument:

Suppose

$$(1 + x_{i_1} t)^{\alpha_1} (1 + x_{i_2} t)^{\alpha_2} \cdots (1 + x_{i_e} t)^{\alpha_e} = 1, \quad \alpha_i \neq 0.$$

Write $\alpha_i = p^{s_i} \beta_i$, $p \nmid \beta_i$. Then

$$(1 + y_{i_1} t^{p^{s_1}})^{\beta_1} (1 + y_{i_2} t^{p^{s_2}})^{\beta_2} \cdots (1 + y_{i_e} t^{p^{s_e}})^{\beta_e} = 1 \text{ with } y_{i_k} = x_{i_k}^{p^{s_k}}.$$

We observe that the coefficient of $y_{i_1} y_{i_2} \cdots y_{i_e} t^{\Sigma p^{s_j}}$ on the left-hand side is $\beta_1 \beta_2 \cdots \beta_s$, which is not zero, giving us a contradiction.

Substituting $1 + x_i t$ for y_i in $w(y_1, \dots, y_m)$ we get an expression

$$1 \neq (1 + x_{i_1} t)^{l_1} (1 + x_{i_2} t)^{l_2} \cdots (1 + x_{i_s})^{l_s} \in R[[t]].$$

This can be rewritten in the form

$$\sum_{i \geq 1} p_i(x_1, \dots, x_m) t^i \neq 0.$$

Thus there exists $l \geq 1$ such that $p_l(x_1, \dots, x_m) \neq 0$ and $p_l(x_1, \dots, x_m) \in R = F\{x_1, \dots, x_m\}$ is a homogeneous polynomial of degree l.

Now, if $r_1, \dots, r_m \in N(FG)$, the elements $1 + r_i\lambda$ are invertible in FG with inverse

$$(1 + r_i\lambda)^{-1} = (1 - r_i\lambda + r_i^2\lambda^2 - \cdots).$$

Hence, by evaluating the group identity on the elements $1 + r_1\lambda, \dots, 1 + r_m\lambda$, we get

$$\sum_{i=1}^{k} p_i(r_1, \dots, r_m)\lambda^i = 0,$$

for some positive integer k and $p_t(r_1, \dots, r_m) = 0$ for all $t > k$. Hence, if $l > k$, then $p_l(r_1, \dots, r_m) = 0$. On the other hand if $l \le k$, since F is infinite there exist nonzero distinct elements $\lambda_1, \dots, \lambda_{k+1} \in F$ such that $\sum_{i=1}^{k} p_i(r_1, \dots, r_m)\lambda_j^i = 0$, for all $j = 1, \dots, k+1$. But then

$$\begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^k \\ 1 & \lambda_2 & \dots & \lambda_2^k \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_{k+1} & \dots & \lambda_{k+1}^k \end{pmatrix} \begin{pmatrix} 0 \\ p_1(r_1, \dots, r_m) \\ \vdots \\ p_k(r_1, \dots, r_m) \end{pmatrix} = 0$$

and, since the determinant of the above Vandermonde matrix is nonzero, we get

$$p_1(r_1, \dots, r_m) = \cdots = p_k(r_1, \dots, r_m) = 0.$$

Thus $p_l(r_1, \dots, r_m) = 0$ also in this case. We have proved that $p_l(x_1, \dots, x_m)$ is a PI for $N(FG)$.

By a standard linearization process it follows that $N(FG)$ satisfies a multilinear polynomial identity:

$$f(x_1, \dots, x_d) = \sum_{\sigma \in S_d} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(d)}$$

where $\alpha_\sigma \in F$ and S_d is the symmetric group of degree d. Since $N(FG)$ is not nilpotent we can choose $a_1, \dots, a_d \in N(FG)$ so that $a_1 a_2 \cdots a_d \neq 0$. Then

$$a_1 FG a_2 FG \cdots a_d FG \neq 0$$

and

$$\sum_{\sigma \in S_d} \alpha_\sigma a_{\sigma(1)} x_{\sigma(1)} a_{\sigma(2)} x_{\sigma(2)} \cdots a_{\sigma(d)} x_{\sigma(d)}$$

is a nondegenerate multilinear generalized polynomial identity for FG. By a theorem of Passman [6, p. 202] we conclude that $(G : \phi) < \infty$ and $(\phi' : 1) < \infty$. It remains to prove that ϕ' is a p-group. This follows because now G is locally finite and we can apply 2.3. □

Corollary. *Let F be an infinite field and let G be a torsion nilpotent group which is nonabelian. Then $U(FG)$ satisfies a group identity $\Leftrightarrow$ FG is Lie n-Engel $\Leftrightarrow$ $(u^{p^m}, v) = 1$ $\forall u, v \in U$ and some fixed $m, p = \operatorname{char} F$.*

Proof. Suppose U satisfies a group identity. Then by the theorem we have $(G : \phi) < \infty$, $(\phi' : 1) < \infty$. Remembering that $G = P \times Q$ is locally finite, we conclude by 2.3 that $G' \subseteq P$. Thus Q is central and $Q \subseteq \phi$. Thus $(G : \phi)$ is a p-power. Moreover, $(\phi' : 1)$ is a p-power. It follows by a theorem of Sehgal [8, p. 155] that FG is Lie n-Engel.

Suppose that FG is Lie n-Engel. Then $[x, \underbrace{y, \dots, y}_{n}] = 0$. Choosing m so that $p^m > n$, we have $0 = [x, \underbrace{y, \dots, y}_{p^m}] = [x, y^{p^m}]$ which implies $(u^{p^m}, v) = 1$ for $u, v \in U$. The remaining implication is trivial. This completes the proof of the Corollary. □

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