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NONCOMMUTATIVE H^2 SPACES

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ABSTRACT. Let $\mathcal{M}$ be a von Neumann algebra with a faithful, finite, normal tracial state τ, and let $\mathcal{A}$ be a finite, maximal subdiagonal algebra of $\mathcal{M}$. Let H^2 be the closure of $\mathcal{A}$ in the noncommutative Lebesgue space $L^2(\mathcal{M}, \tau)$. Then H^2 possesses several of the properties of the classical Hardy space on the circle, including a commutant lifting theorem, some results on Toeplitz operators, an H^1 factorization theorem, Nehari's Theorem, and harmonic conjugates which are L^2 bounded.

1. INTRODUCTION

In [1] Arveson introduced the concept of a subdiagonal algebra in order to unify the analysis of several broad classes of nonselfadjoint operator algebras. Arveson drew a close analogy between a subdiagonal algebra and the classical Hardy space H^∞, the boundary values of the bounded analytic functions on the disk. Roughly, a subdiagonal algebra $\mathcal{A}$ stands in relation to its von Neumann algebra $\mathcal{M}$ as H^∞ stands in relation to the Lebesgue space L^∞ of the unit circle. Subsequently, several authors studied the invariant subspaces of $\mathcal{A}$ acting on the noncommutative Lebesgue space L^p [6], [8], [10]. There has also been considerable investigation of analytic crossed products, which are a type of subdiagonal algebra introduced by McAsey, Muhly and Saito, including their invariant subspace structure [8], [10], maximality among weak* closed subalgebras of $\mathcal{M}$ [8], associated Toeplitz operators [11] and Hankel operators [5]. We shall study the closure of $\mathcal{A}$ in L^2 as an analogue of the classical Hardy space H^2, and so obtain analogues of several classical results including a commutant lifting theorem, some results on Toeplitz operators, an H^1 factorization theorem, Nehari's Theorem on the norm of a Hankel operator, and the existence and L^2 boundedness of the harmonic conjugate.

Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal finite tracial state τ. For $1 \le p < \infty$, let $L^p = L^p(\mathcal{M}, \tau)$ denote the noncommutative Lebesgue space which is associated with $\mathcal{M}$ and τ (cf. [3], [9], [12]). For $t \in \mathcal{M}, x \in L^2$, let $L_t(x) = tx$ and $R_t(x) = xt$. Then $\mathcal{L} = \{L_t\colon\ t \in \mathcal{M}\}$ and $\mathcal{R} = \{R_t\colon\ t \in \mathcal{M}\}$ are von Neumann algebras on L^2 which are each other's commutants. Furthermore, the map $t \to L_t$ (resp. $t \to R_t$) is a normal, *-isomorphism (resp. *-anti-isomorphism) of $\mathcal{M}$ onto $\mathcal{L}$ (resp. $\mathcal{R}$), and the identity 1 is a cyclic and separating vector for $\mathcal{L}$ and $\mathcal{R}$. The map $x \to x^*$ on $\mathcal{M}$ extends to a conjugate linear isometry on L^p. As is customary, we identify $\mathcal{M}$ with L^∞ while the ultraweak topology on $\mathcal{M}$ will

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be identified with the weak* topology on L^∞ regarded as the dual of L^1. We now introduce $\mathcal{A}$, the noncommutative analogue of H^∞ (cf. [1], [8]).

Definition. Let $\mathcal{A}$ be a w*-closed unital subalgebra of $\mathcal{M}$, and let Φ be a faithful, normal expectation from $\mathcal{M}$ onto the diagonal $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$. Then $\mathcal{A}$ is a finite, maximal subdiagonal subalgebra of $\mathcal{M}$ with respect to Φ if:

(1) $\mathcal{A} + \mathcal{A}^*$ is w*-dense in $\mathcal{M}$,
(2) $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in \mathcal{A}$,
(3) $\mathcal{A}$ is maximal among those subalgebras satisfying (1) and (2), and
(4) $\tau \circ \Phi = \tau$.

For $\mathcal{S} \subset L^p$, $1 \le p < \infty$, let $[\mathcal{S}]_p$ denote the closure of $\mathcal{S}$ in L^p. Let $H^p = [\mathcal{A}]_p$, $H_0^p = [\{x \in \mathcal{A} \colon \Phi(x) = 0\}]_p$, and P be the orthogonal projection of L^2 onto H^2. Then Φ extends to the orthogonal projection of L^2 onto $[\mathcal{D}]_2$ and $L^2 = H^2 \oplus (H_0^2)^* = H_0^2 \oplus [\mathcal{D}]_2 \oplus (H_0^2)^*$ by [8, Proposition 1.1]. For $t \in \mathcal{M}$ we define the (left) Toeplitz operator with symbol t by $T_t = PL_tP$. We define the (left) Hankel operator with symbol t by $H_t = (1-P)L_tP$.

2. A commutant lifting theorem

Let $\mathcal{T}$ denote the algebra $\{L_a \colon a \in \mathcal{A}\}$ considered as an algebra of operators on H^2. In this section we identify $\mathcal{T}'$, the operators on H^2 that commute with $\mathcal{T}$. In particular, we show that every element of $\mathcal{T}'$ lifts to an element of $\mathcal{R}$. Our result is analogous to Yoshino's Theorem [13], which describes the commutant of a rationally cyclic subnormal operator. Indeed, if μ is a measure on the complex plane with compact support K, R is the algebra of rational functions with poles off of K, R^2 is the $L^2(\mu)$ closure of R, and S is multiplication by z restricted to R^2, then the commutant of S is $\{M_\phi \colon \phi \in R^2 \cap L^\infty(\mu)\}$, where $M_\phi f = \phi f$ is the usual multiplication operator. It follows that if X is an operator on R^2 which commutes with multiplications by elements of R, then $X = M_\phi$ for some $\phi \in L^\infty(\mu)$ with $M_\phi(R^2) \subset R^2$. So the commutant of R acting on its $L^2(\mu)$ closure lifts to the von Neumann algebra $L^\infty(\mu)$ acting on $L^2(\mu)$. For $\mathcal{T}$ acting on H^2 we have the following result.

Theorem 1. *If $X \in \mathcal{T}'$, then there exists $b \in \mathcal{M}$ such that H^2 is invariant for R_b, $X = R_b$ on H^2, and $\|X\| = \|b\|$.*

Proof. Let $X(1) = h$. First, we will show that $\|rh\|_2 \le \|X\|\,\|r\|_2$ for all $r \in \mathcal{M}$. Let $\varepsilon > 0$. Then $r^*r + \varepsilon 1$ is an invertible positive operator in $\mathcal{M}$. So by [1, Corollary 4.2.4] there is an $a \in \mathcal{A}$ such that $r^*r + \varepsilon 1 = a^*a$. Thus

$$\begin{aligned}\langle (r^*r + \varepsilon 1)h, h\rangle &= \langle a^*ah, h\rangle = \|ah\|_2^2 = \|Xa\|_2^2 \\ &\le \|X\|^2\|a\|_2^2 = \|X\|^2\langle a^*a, 1\rangle = \|X\|^2\langle r^*r + \varepsilon 1, 1\rangle.\end{aligned}$$

Letting ε go to 0, we obtain $\|rh\|_2^2 \le \|X\|^2\|r\|_2^2$.

Define a map Y via $Y(r) = rh$. Clearly Y extends to a bounded operator on L^2 which commutes with $\mathcal{L}$, and $\|Y\| \le \|X\|$. Thus there is a $b \in \mathcal{M}$ such that $Y = R_b$. Because $h \in H^2$, Y leaves H^2 invariant. Obviously Y agrees with X on H^2, and so $\|b\| = \|Y\| = \|X\|$.

3. Factorizations, Toeplitz and Hankel operators

Our next theorem is a useful factorization result for elements of L^2, which might be expressed as $L^2 = L^\infty H^2 = H^2 L^\infty$. The proof in the classical case is an application of the fact that if $\log|f|$ is integrable for an L^2 function f, then $|f| = |h|$ for some H^2 function h. However, this fact is a consequence of Szegő's Theorem, and it is not yet known if Szegő's Theorem extends to the subdiagonal algebra setting. Nevertheless [10, Proposition 1] is a sufficient substitute. As consequences we obtain some facts about Toeplitz operators, the factorization of elements of H^1 as products of H^2 elements (previously obtained by Saito), and Nehari's Theorem.

Theorem 2. *For every $\varepsilon > 0$ and $z \in L^2$, there exist $h_1, h_2 \in H^2$ and $v_1, v_2 \in \mathcal{M}$, such that $z = v_1 h_1 = h_2 v_2$, $\|v_i\| \le 1$, $\|h_i\|_2 < (1+\varepsilon)\|z\|_2$, and $h_i^{-1} \in \mathcal{A}$ for $i = 1, 2$.*

Proof. We prove the existence of v_1 and h_1. The proof for v_2 and h_2 is similar. Choose $\delta < \sqrt{2\varepsilon + \varepsilon^2}\|z\|_2$. Consider the positive weak* continuous linear functional $\omega_{\delta 1} + \omega_z$ on $\mathcal{R}$. By [7, Theorem 7.2.3] there is a vector $y \in L^2$ such that $\omega_y = \omega_{\delta 1} + \omega_z$ on $\mathcal{R}$. Clearly, $\omega_{\delta 1} \le \omega_y$ and $\omega_z \le \omega_y$ on $\mathcal{R}$. It follows that there exist $r, s \in \mathcal{M}$ such that $\delta 1 = ry$ and $z = sy$. Because $\mathcal{M}$ is finite, $\delta^{-1} r$ has an inverse in L^2.

By [10, Proposition 1] there exist a unitary $u \in \mathcal{M}$ and $g \in \mathcal{A}$ such that $\delta^{-1} r = gu$, and $h = g^{-1} \in H^2$. Thus $y = u^* h, z = su^* h$, and $\|h\|_2 = \|y\|_2 = \sqrt{\delta^2 + \|z\|_2^2} < (1+\varepsilon)\|z\|_2$. Now let $v_1 = su^*$ and $h_1 = h$.

In the classical setting, one has that the map $t \to T_t$ is an isometry on L^∞, but the proof relies heavily on the fact that L^∞ is commutative. We do not know if this map is isometric in our noncommutative setting. We do have the following partial results, including the fact that the map is very well behaved on $\mathcal{A}$.

Corollary 3. *For every $t \in \mathcal{M}$, $\|L_t P\| = \|t\|$.*

Proof. We have $\|L_t\| = \|t\|$. So given $\varepsilon > 0$, choose z such that $\|z\|_2 < 1$ and $\|tz\|_2 \ge \|t\| - \varepsilon$. By Theorem 2 we can write $z = hv$ where $v \in \mathcal{M}$, $\|v\| \le 1$, $h \in H^2$, and $\|h\|_2 < 1$. Thus $\|tz\|_2 = \|thv\|_2 \le \|th\|_2$, so $\|th\|_2 \ge \|t\| - \varepsilon$.

Corollary 4. *For every $t \in \mathcal{M}$, $\|T_{t^*t}\| = \|t^*t\|$.*

Proof. We have $\|T_{t^*t}\| = \|PL_{t^*t}P\| = \|PL_t^* L_t P\| = \|L_t P\|^2 = \|t\|^2 = \|t^*t\|$.

Corollary 5. *The map $a \to T_a$ is an isometric algebra isomorphism and a weak* homeomorphism of $\mathcal{A}$ onto $\{T_a : a \in \mathcal{A}\}$.*

Proof. For $a \in \mathcal{A}$, $T_a = L_a P$, so by Corollary 3 the map is an isometry. Because the map $a \to L_a$ is a weak* continuous isomorphism and H^2 is invariant for $\{L_a : a \in \mathcal{A}\}$, the map $a \to T_a$ is a weak* continuous isomorphism. It follows from [2, Theorem 1.20] that $\{T_a : a \in \mathcal{A}\}$ is weak* closed and the map is a weak* homeomorphism.

Let $\mathcal{A}_*$ denote the space of weak* continuous linear functionals on $\mathcal{A}$. The next result shows that the weak* continuous functionals on the algebra $\{T_a : a \in \mathcal{A}\}$ have a rank one structure. In the language of the theory of dual algebras [2] the algebra has property $\mathbf{A}_1(1)$.

Corollary 6. *For every $\varepsilon > 0$ and for every $\phi \in \mathcal{A}_*$, there exist $g, h \in H^2$ such that $\phi(a) = \langle ag, h\rangle$ for $a \in \mathcal{A}$, and $\|g\|_2\|h\|_2 < (1+\varepsilon)\|\phi\|$.*

Proof. For $\varepsilon > 0$, choose $\eta > 0$ such that $(1+\eta)^2 < 1 + \varepsilon$. There is a functional $\hat{\phi}$ in $\mathcal{M}_*$ such that $\hat{\phi}$ extends ϕ and $\|\hat{\phi}\| < (1+\eta)\|\phi\|$ by [4, Lemma 2.4]. Because of the duality between L^1 and $\mathcal{M}$, there is an $f \in L^1$ such that $\hat{\phi}(t) = \tau(tf)$ for $t \in \mathcal{M}$ and $\|\hat{\phi}\| = \|f\|_1$. So there exist vectors $x, y \in L^2$ such that $f = xy$ and

$\|x\|_2\|y\|_2 = \|f\|_1 = \|\hat{\phi}\| < (1+\eta)\|\phi\|$. By Theorem 2 $x = gv$ with $g \in H^2$, $v \in \mathcal{M}$, $\|v\| \leq 1$, and $\|g\|_2 < (1+\eta)\|x\|_2$. Thus, for all $a \in \mathcal{A}$,

$$\phi(a) = \langle L_a gv, y\rangle = \langle L_a g, yv^*\rangle = \langle L_a g, P(yv^*)\rangle.$$

Let $h = P(yv^*)$, and the result follows.

We now proceed to establish an analogue of the factorization theorem for H^1. Recall that any function f in the classical H^1 space can be written $f = gh$ where g, h are in H^2 and $\|f\|_1 = \|g\|_2\|h\|_2$. The following result was previously obtained by Saito [11, Lemma 5.5], but we include it for completeness.

Corollary 7. *For every $\varepsilon > 0$ and for every $f \in H^1$, there exist $g, h \in H^2$ such that $f = gh$ and $\|g\|_2\|h\|_2 < (1+\varepsilon)\|f\|_1$. If $f \in H^1_0$, then h is in H^2_0.*

Proof. Because $f \in L^1$, there exist vectors $x, y \in L^2$ such that $f = xy$ and $\|x\|_2\|y\|_2 = \|f\|_1$. By Theorem 2 we can find $g \in H^2$ and $v \in \mathcal{M}$ such that $x = gv$, $\|v\| \leq 1$, $\|g\|_2 < (1+\varepsilon)\|x\|_2$ and $g^{-1} \in \mathcal{A}$. Now by [10, Lemma 3] $H^1 = \{x \in L^1 \colon \tau(xy) = 0 \text{ for all } y \in H^\infty_0\}$. Thus, for $a \in \mathcal{A}_0$,

$$\tau(af) = \tau(axy) = \tau(agvy) = \langle ag, (vy)^*\rangle.$$

Let $h = vy$. Then $h^* \in [\mathcal{A}_0 g]_2^\perp$. But $[\mathcal{A}_0 g]_2 = [\mathcal{A}_0]_2$, because $g^{-1} \in \mathcal{A}$. Thus $h^* \in [\mathcal{A}_0]_2^\perp = (H^2)^*$, so $h \in H^2$ and $f = xy = gvy = gh$. Finally, $\|g\|_2\|h\|_2 < (1+\varepsilon)\|x\|_2\|y\|_2 = (1+\varepsilon)\|f\|_1$.

Note that if $f \in H^1_0$, then $\tau(af) = 0$ for all $a \in \mathcal{A}$. So $h^* \in (H^2)^\perp = (H^2_0)^*$, and thus $h \in H^2_0$.

As a consequence of the previous factorization theorem, we can obtain an exact analogue of Nehari's Theorem on the norm of a Hankel operator. In the classical setting, the norm of a Hankel operator H_f is given by the distance in L^∞ from the symbol f to H^∞. This remains true in the noncommutative setting.

Corollary 8. *For every $t \in \mathcal{M}$, $\|H_t\| = \operatorname{dist}(t, \mathcal{A})$.*

Proof. By [10, Lemma 4] $H^1_0 = \{f \in L^1 \colon \tau(fy) = 0 \text{ for all } y \in \mathcal{A}\}$. By Corollary 7 $\{f \in H^1_0 \colon \|f\|_1 < 1\} = \{gh \colon g \in H^2, h \in H^2_0, \|g\|_2\|h\|_2 < 1\}$. Because $\mathcal{A}$ is w*-closed,

$$\begin{aligned}
\operatorname{dist}(t, \mathcal{A}) &= \sup\{|\tau(tf)| \colon f \in L^1, \tau(yf) = 0 \text{ for all } y \in \mathcal{A}, \|f\|_1 < 1\} \\
&= \sup\{|\tau(tf)| \colon f \in H^1_0, \|f\|_1 < 1\} \\
&= \sup\{|\tau(tgh)| \colon g \in H^2, h \in H^2_0, \|g\|_2\|h\|_2 < 1\} \\
&= \sup\{|\langle L_t g, h^*\rangle| \colon g \in H^2, h \in H^2_0, \|g\|_2\|h\|_2 < 1\} \\
&= \|H_t\|.
\end{aligned}$$

4. Harmonic conjugates

Every harmonic function u on the unit disk has an associated harmonic function $\tilde{u}$, the harmonic conjugate of u, such that $u + i\tilde{u}$ is analytic and $\tilde{u}(0) = 0$. The map $u \to \tilde{u}$ is a real linear transformation which is L^2 bounded. We will establish an analogue of this result by thinking of Re $\mathcal{A}$, the real parts of the operators in $\mathcal{A}$, as a noncommutative version of the space of bounded harmonic functions. We first construct a real linear map from Re $\mathcal{A}$ to $\mathcal{A}$ which is the analogue of the Herglotz transform, i.e. integration against the kernel $(e^{i\theta} + z)(e^{i\theta} - z)^{-1}$.

Theorem 9. *There is a real linear transformation $T\colon \mathrm{Re}\,\mathcal{A} \to \mathcal{A}$ such that $u = \mathrm{Re}(Tu)$, $\Phi(\mathrm{Im}\,Tu) = 0$, and $\|Tu\|_2 \le \sqrt{2}\|u\|_2$ for all $u \in \mathrm{Re}\,\mathcal{A}$. Thus T extends to a bounded operator from $[\mathrm{Re}\,\mathcal{A}]_2$ to H^2.*

Proof. Let $u \in \mathrm{Re}\,\mathcal{A}$. So $u = \mathrm{Re}\,g$ for some $g \in \mathcal{A}$. Let $a = g - \frac{1}{2}\Phi(g - g^*)$. Then $a \in \mathcal{A}$, $u = \mathrm{Re}\,a$, and $\Phi(\mathrm{Im}\,a) = 0$. Thus there exists an $a \in \mathcal{A}$ with the desired properties. We now show that such an element of $\mathcal{A}$ is unique. Suppose that $u = \mathrm{Re}\,a = \mathrm{Re}\,h$ for $a, h \in \mathcal{A}$, and $\Phi(\mathrm{Im}\,a) = \Phi(\mathrm{Im}\,h) = 0$. Then $a + a^* = h + h^*$ implies that $a - h = h^* - a^*$, so $(a-h)^* = h - a$. Furthermore, $\Phi(\mathrm{Im}\,a) = \Phi(\mathrm{Im}\,h) = 0$ implies that $\Phi(a) = \Phi(a^*)$ and $\Phi(h) = \Phi(h^*)$. Consequently, $\Phi(a) = \Phi(h)$, because $\mathrm{Re}\,a = \mathrm{Re}\,h$. Thus

$$\Phi((a-h)(a-h)^*) = \Phi((a-h)(h-a)) = \Phi(a-h)\Phi(h-a) = 0.$$

So $a = h$, because Φ is faithful. Thus we can define $Tu = a$, where a is the unique element of $\mathcal{A}$ with $u = \mathrm{Re}\,a$ and $\Phi(\mathrm{Im}\,a) = 0$. It is easy to see that T is real linear.

We now proceed to establish the inequality $\|Tu\|_2 \le \sqrt{2}\|u\|_2$. Let $Tu = d + b$, where $d \in \mathcal{D}$, $b \in \mathcal{A}_0$. Now $\Phi(\mathrm{Im}\,Tu) = 0$ and $\Phi(b) = \Phi(b^*) = 0$, so we have $d = \Phi(d) = \Phi(d^*) = d^*$. Thus

$$\|Tu\|_2^2 = \tau((d+b)(d+b^*)) = \|d\|_2^2 + \tau(db^*) + \tau(db) + \|b\|_2^2 = \|d\|_2^2 + \|b\|_2^2.$$

By a simple computation we have

$$0 = \tau(b^2) = \tau((\mathrm{Re}\,b)^2) - \tau((\mathrm{Im}\,b)^2) + 2i\tau((\mathrm{Re}\,b)(\mathrm{Im}\,b)).$$

Note that

$$\overline{\tau((\mathrm{Re}\,b)(\mathrm{Im}\,b))} = \tau((\mathrm{Im}\,b)(\mathrm{Re}\,b)) = \tau((\mathrm{Re}\,b)(\mathrm{Im}\,b)).$$

By equating real and imaginary parts we obtain

$$\|(\mathrm{Re}\,b)\|_2^2 = \|(\mathrm{Im}\,b)\|_2^2 \quad \text{and} \quad \tau((\mathrm{Re}\,b)(\mathrm{Im}\,b)) = 0.$$

So

$$\|b\|_2^2 = \|(\mathrm{Re}\,b)\|_2^2 + \|(\mathrm{Im}\,b)\|_2^2 = 2\|(\mathrm{Re}\,b)\|_2^2.$$

Thus

$$\|Tu\|_2^2 = \|\mathrm{Re}\,d\|_2^2 + 2\|\mathrm{Re}\,b\|_2^2.$$

Now because b is orthogonal to $\mathcal{D}$, we have $\tau((\mathrm{Re}\,d)(\mathrm{Re}\,b)) = 0$. Consequently,

$$\|Tu\|_2^2 \le 2(\|\mathrm{Re}\,d\|_2^2 + \|\mathrm{Re}\,b\|_2^2) = 2\|\mathrm{Re}(d+b)\|_2^2 = 2\|u\|_2^2.$$

Corollary 10. *For each $u \in [\mathrm{Re}\,\mathcal{A}]_2$ there is a unique $\tilde{u}$ in $[\mathrm{Re}\,\mathcal{A}]_2$ such that $u + i\tilde{u} \in H^2$ and $\Phi(\tilde{u}) = 0$. Furthermore, $\|\tilde{u}\|_2 \le \sqrt{2}\|u\|_2$.*

Proof. Let $\tilde{u} = \mathrm{Im}\,Tu$.

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