

l_∞ AND INTERPOLATION BETWEEN BANACH LATTICES

NAHUM ZOBIN AND VERONICA ZOBIN

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We study the possibility of obtaining the l_∞ -norm by an interpolation method starting from a couple of Banach lattice norms. We describe all couples of Banach lattice norms in \mathbb{R}^n such that the l_∞ -norm is a strict interpolation norm between them. Further we consider the possibility of obtaining the l_∞ -norm by any method which guarantees interpolation of not only linear operators (= bilinear forms on $\mathbb{R}^n \times \mathbb{R}^n$) but also of all polylinear forms. Here we show that either one of the initial norms has to be proportional to the l_∞ -norm, or both have to be weighted l_∞ -norms.

INTRODUCTION

The problem discussed in this article was inspired by a question posed by N. Kalton (Workshop on Interpolation spaces, Haifa, 1990), which he formulated as follows: is l_∞ a black hole? He noticed that it is impossible to obtain the space l_∞ by the complex method construction starting from a wide range of Banach couples both non-isometric to l_∞ (including, in particular, couples of Banach lattices). He asked whether this result holds for any Banach couple. We modify the problem allowing more general interpolation methods but restricting ourselves to couples of Banach lattices.

So we are investigating the following problem:

given a couple of Banach lattices, is it possible to obtain l_∞ as a strict interpolation space between them ?

Simple examples show that this is possible.

Example. Let $V = \mathbb{R}^2$,

$$U_0 = \{(x_1, x_2) \in V : |x_1| \leq 1, |x_1| + |x_2| \leq 2\},$$

$$U_1 = \{(x_1, x_2) \in V : |x_2| \leq 1, |x_1| + |x_2| \leq 2\}.$$

Obviously U_0 and U_1 are unit balls of Banach lattice norms and the unit ball of l_∞^2 is the intersection of U_0 and U_1 , so l_∞^2 is a strict interpolation space between these Banach lattices.

Received by the editors December 1, 1994 and, in revised form, September 13, 1995.

1991 *Mathematics Subject Classification.* Primary 46M35.

Key words and phrases. Banach lattices, interpolation of operators, complex method.

The research of the first author was partially supported by grants from the Ministry of Absorption, the Ministry of Science and Technology (Israel) and by the Rashi Foundation (France-Israel). The research of the second author was partially supported by a grant from the Ministry of Science, Israel, and “Maagara”—a special project for absorption of new immigrants—at the Department of Mathematics, Technion, Haifa, Israel.

We shall see that there exist quite a few couples of Banach lattice norms such that the l_∞ space is a strict interpolation space between them. Let us note that the spaces constructed by the complex method have much better properties than general interpolation spaces, namely, they allow us to interpolate not only linear but also polylinear operators.

Let us recall several basic definitions (see, e.g., [1]).

So, let $V = \mathbb{R}^n$. Let G denote the group of operators acting in V as follows:

$$(x_1, \dots, x_n) \rightarrow (\varepsilon_1 x_1, \dots, \varepsilon_n x_n), \quad \varepsilon_i = \pm 1.$$

A norm in V is called a **Banach lattice norm** if it is G -invariant.

We consider the natural scalar product on V :

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i.$$

For every norm $\|\cdot\|_\alpha$ on the space V let $\|\cdot\|^\alpha$ denote the dual norm on $V = V'$, i.e.,

$$\|y\|^\alpha = \sup\{\langle x, y \rangle : \|x\|_\alpha \leq 1\}.$$

One can easily show that the dual of a Banach lattice norm is a Banach lattice norm itself.

A norm $\|\cdot\|_!$ on V is called a (**strict**) **interpolation norm** for a couple of norms $\{\|\cdot\|_i : i = 0, 1\}$ on V (notation: $\|\cdot\|_! \in \text{Int}_1(\|\cdot\|_i : i = 0, 1)$) if for every linear operator $L : V \rightarrow V$ the following estimate holds:

$$\sup\{\|Lx\|_! : \|x\|_! \leq 1\} \leq \max_{i=0,1} \sup\{\|Lx\|_i : \|x\|_i \leq 1\}$$

or, in other words, for every bilinear form $f : V \times V' \rightarrow \mathbb{R}$ the following estimate holds:

$$\sup\{|f(x, y)| : \|x\|_!, \|y\|^\dagger \leq 1\} \leq \max_{i=0,1} \sup\{|f(x, y)| : \|x\|_i, \|y\|^\dagger \leq 1\}.$$

First we are going to give an answer to the following

PROBLEM 1. Describe all couples of Banach lattice norms in \mathbb{R}^n for which the l_∞^n -norm is a strict interpolation norm.

Next we study the situation with interpolation of polylinear forms.

Definition. Let $\|\cdot\|_!$, $\|\cdot\|_0$, $\|\cdot\|_1$ be three norms on a linear space V . We say that the norm $\|\cdot\|_!$ is a (**strict**) (k, m) -**interpolation norm** between the norms $\|\cdot\|_i$ ($i = 0, 1$) (notation: $\|\cdot\|_! \in (k, m)\text{-Int}_1(\|\cdot\|_0, \|\cdot\|_1)$) if for every polylinear form

$$f : V \times V \times \dots \times V \times V' \times V' \times \dots \times V' \rightarrow \mathbb{R}$$

(k copies of V and m copies of V') the following inequality holds:

$$N_!(f) \leq \max\{N_i(f) : i = 0, 1\}$$

where

$$N_i(f) := \sup\{|f(x_1, \dots, x_k; y_1, \dots, y_m)| : \|x_j\|_i, \|y_j\|^\dagger \leq 1\}.$$

We say that the norm $\|\cdot\|_!$ is a (**strict**) ω -**interpolation norm** between the norms $\|\cdot\|_i$ ($i = 0, 1$) (notation: $\|\cdot\|_! \in \omega\text{-Int}_1(\|\cdot\|_0, \|\cdot\|_1)$) if $\|\cdot\|_! \in (k, m)\text{-Int}_1(\|\cdot\|_0, \|\cdot\|_1)$ for every $k, m \in \mathbb{N}$.

So the usual interpolation norm is simply a (1,1)-interpolation norm.

It is well known that the norms constructed by the complex method are (strict) ω -interpolation norms between the initial couple of norms.

PROBLEM 2. Describe the set of couples of Banach lattice norms on \mathbb{R}^n , for which the l_∞^n norm is an ω -interpolation norm.

We show that each such couple either consists of two norms isometric to the l_∞^n -norm, or one of these norms is proportional to the l_∞^n -norm. So the effect noticed by N. Kalton for the complex method applied to a couple of Banach lattices is really a consequence of the ω -interpolation.

We apply an approach developed earlier for closely related problems of the Interpolation Theory, see, e.g., [4].

Solution of Problem 1. Consider the following norms on $V = \mathbb{R}^n : \|\cdot\|_\infty$ is the l_∞^n -norm, $\|\cdot\|_i$ ($i = 0, 1$) are a couple of Banach lattice norms. We are going to describe all couples of norms $\|\cdot\|_i$ ($i = 0, 1$), such that $\|\cdot\|_\infty \in \text{Int}_1(\|\cdot\|_i : i = 0, 1)$.

Consider the tensor product space $V \otimes V$. It is in a natural duality with the space $\text{End } V$ of linear operators on V :

$$\langle \langle \sum_i x^i \otimes f^i, L \rangle \rangle := \sum_i \langle Lx^i, f^i \rangle \quad (x^i, f^i \in V).$$

This duality permits to reformulate the definition of an interpolation norm in dual terms:

Assertion 1 (see [3, 4]). $\|\cdot\|_! \in \text{Int}_1(\|\cdot\|_0, \|\cdot\|_1)$ if and only if

$$\forall x, y \in V, \quad \|x\|_! \|y\|_! \geq \min \left\{ \sum_{i,k} \|x_{ik}\|_k \|y_{ik}\|_k^k : x \otimes y = \sum_{i,k} x_{ik} \otimes y_{ik} \right\}.$$

In our case it is sufficient (and, certainly, necessary) to guarantee that for every $j = 1, \dots, n$

$$1 \geq \min \left\{ \sum_{i,k} \|x_{ik}\|_k \|y_{ik}\|_k^k : \mathbb{I} \otimes e_j = \sum_{i,k} x_{ik} \otimes y_{ik} \right\}$$

where $\mathbb{I} = (1, 1, \dots, 1)$, $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ (1 on the j -th place). This follows from the fact that the set of extreme points of the unit ball of $\|\cdot\|_\infty$ consists of the vectors $g\mathbb{I}$ ($g \in G$), and the set of extreme points of the unit ball of $\|\cdot\|^\infty$ consists of the vectors $\pm e_j$, $j = 1, \dots, n$.

So, for every $j = 1, 2, \dots, n$ there must exist a_{jik}, f_{jik} , such that

$$(*) \quad \mathbb{I} \otimes e_j = \sum_{i,k} a_{jik} \otimes f_{jik}$$

and

$$1 \geq \sum_{i,k} \|a_{jik}\|_k \|f_{jik}\|_k^k.$$

Let G_j denote the following subgroup of G :

$$G_j = \{T \in G : Te_j = e_j\}.$$

Let us average (*) with respect to the group of operators

$$\{id \otimes T : T \in G_j\}.$$

We obtain

$$\mathbb{I} \otimes e_j = \sum_{i,k} a_{jik} \otimes (\text{Card } G_j)^{-1} \left(\sum_{T \in G_j} T f_{jik} \right).$$

One can easily see that

$$(\text{Card } G_j)^{-1} \sum_{T \in G_j} T f = \langle f, e_j \rangle e_j.$$

Moreover, since the norms $\|\cdot\|^k$, $k = 0, 1$, are G -invariant then

$$\|\langle f, e_j \rangle e_j\|^k = \|(\text{Card } G_j)^{-1} \sum_{T \in G_j} T f\|^k \leq \|f\|^k.$$

So, we obtain

$$\mathbb{I} \otimes e_j = \sum_{i,k} a_{jik} \otimes \langle f_{jik}, e_j \rangle e_j$$

and

$$1 \geq \sum_{i,k} \|a_{jik}\|_k |\langle f_{jik}, e_j \rangle| \|e_j\|^k.$$

Put $a_{jk} = \sum_i a_{jik} \langle f_{jik}, e_j \rangle$. Obviously, $\|a_{jk}\|_k \leq \sum_i \|a_{jik}\|_k |\langle f_{jik}, e_j \rangle|$. In other words, for every $j = 1, 2, \dots, n$, there exist a_{jk} , $k = 0, 1$, such that

$$\mathbb{I} = a_{j0} + a_{j1},$$

and

$$1 \geq \|a_{j0}\|_0 \|e_j\|^0 + \|a_{j1}\|_1 \|e_j\|^1.$$

Note that

$$\|a_{j0}\|_0 \|e_j\|^0 + \|a_{j1}\|_1 \|e_j\|^1 \geq \langle a_{j0}, e_j \rangle + \langle a_{j1}, e_j \rangle = \langle \mathbb{I}, e_j \rangle = 1.$$

Combining all this, we obtain the following result:

Theorem 1. *The norm $\|\cdot\|_\infty$ is a strict interpolation norm between the Banach lattice norms $\|\cdot\|_i$ ($i = 0, 1$) on \mathbb{R}^n if and only if for every $j = 1, \dots, n$*

$$1 = \min \{ \|a_0\|_0 \|e_j\|^0 + \|a_1\|_1 \|e_j\|^1 : \mathbb{I} = a_0 + a_1 \}.$$

This result may be reformulated in geometric terms.

Let U_i denote the unit ball of the norm $\|\cdot\|_i$, $i = 0, 1$. Let $\Lambda_j = \{x : |\langle x, e_j \rangle| \leq 1\}$. Let $\lambda_{ij} = \max \{ \lambda : \lambda U_i \subseteq \Lambda_j \} = (\|e_j\|^i)^{-1}$. Consider the following set

$$\text{trace}_j U_i = \lambda_{ij} U_i \cap \{x : \langle x, e_j \rangle = 1\}.$$

Note that for every j

$$\mathbb{I} \in \{x : \langle x, e_j \rangle = 1\}.$$

So Theorem 1 is equivalent to the following result:

Theorem 1'. *The norm $\|\cdot\|_\infty$ is a strict interpolation norm between the Banach lattice norms $\|\cdot\|_i$ ($i = 0, 1$) on \mathbb{R}^n if and only if for every $j = 1, \dots, n$*

$$\mathbb{I} \in \text{conv}(\text{trace}_j U_0 \cup \text{trace}_j U_1).$$

Solution of Problem 2. First we need to reformulate the problem in dual terms. This was done, e.g., in the thesis of L. Veselova [2], a student of the first author.

Assertion 2 (see [2]). $\|\cdot\|_! \in (k, m)\text{-Int}_1(\|\cdot\|_0, \|\cdot\|_1)$ if and only if the following estimate holds:

$$\forall x_\kappa \in V \ (\kappa = 1, \dots, k), \ \forall y_\mu \in V \ (\mu = 1, \dots, m)$$

$$\begin{aligned} & \prod_{\kappa} \|x_\kappa\|_! \prod_{\mu} \|y_\mu\|_! \\ & \geq \min \left\{ \sum_{i,j} \|a_{ij}\|_{\otimes j} \|b_{ij}\|^{\otimes j} : x_1 \otimes \dots \otimes x_k \otimes y_1 \otimes \dots \otimes y_m = \sum_{i,j} a_{ij} \otimes b_{ij} \right\}. \end{aligned}$$

Here $a_{ij} \in V^{\otimes k}$, $b_{ij} \in V^{\otimes m}$, $\|\cdot\|_{\otimes j}$, $\|\cdot\|^{\otimes j}$ denote the projective norms on $V^{\otimes k}$, $V^{\otimes m}$, generated by the initial norms $\|\cdot\|_j$, $\|\cdot\|^j$ on V .

Again it is sufficient (and certainly necessary) to consider only the situation when all $x_\kappa = \mathbb{I}$ and $y_\mu = e_{j(\mu)}$. So the condition of (k, m) -interpolation is now equivalent to the estimate

$$\forall e_{j(\mu)} \ (\mu = 1, \dots, m)$$

$$1 \geq \min \left\{ \sum_{i,j} \|a_{ij}\|_{\otimes j} \|b_{ij}\|^{\otimes j} : \mathbb{I}^{\otimes k} \otimes e_{j(1)} \otimes \dots \otimes e_{j(m)} = \sum_{i,j} a_{ij} \otimes b_{ij} \right\}.$$

Again averaging with respect to the subgroup of operators $\{id^{\otimes k} \otimes T_1 \otimes \dots \otimes T_m : T_\mu \in G_{j(\mu)}\}$ we obtain that the condition of (k, m) -interpolation in our case is equivalent to the inequalities:

$$\forall J = \{j_1, \dots, j_m\},$$

$$\begin{aligned} 1 \geq \min \{ & \|a_0(J)\|_{\otimes 0} J^0 + \|a_1(J)\|_{\otimes 1} J^1 : \mathbb{I}^{\otimes k} = a_0(J) + a_1(J), \ a_i(J) \in V^{\otimes k}, \\ & J^i = \prod_{\mu} \|e_{j_\mu}\|^i, \ i = 0, 1\}. \end{aligned}$$

Let us take $k = m$. Note that $\|a\|_{\otimes i} J^i \geq \langle a, \bigotimes_{\mu} e_{j_\mu} \rangle_{\otimes}$, where $\langle \cdot, \cdot \rangle_{\otimes}$ denotes the canonical pairing on $V^{\otimes m}$, arising from the initial scalar product $\langle \cdot, \cdot \rangle$. Therefore

$$\sum_{i=0,1} \|a_i(J)\|_{\otimes i} J^i \geq \sum_{i=0,1} \langle a_i(J), \bigotimes_{\mu} e_{j_\mu} \rangle_{\otimes} = \langle \mathbb{I}^{\otimes m}, \bigotimes_{\mu} e_{j_\mu} \rangle_{\otimes} = 1$$

so the above inequality has to be an equality. Note that the order of factors in the product $\bigotimes_{\mu} e_{j_\mu}$ may be arbitrarily changed.

We have proved the following result:

Theorem 2. $\|\cdot\|_\infty \in (m, m)\text{-Int}_1(\|\cdot\|_0, \|\cdot\|_1)$ if and only if

$$\forall J = \{j_1, \dots, j_m\},$$

$$1 = \inf \{ \|a_0(J)\|_{\otimes 0} J^0 + \|a_1(J)\|_{\otimes 1} J^1 : \mathbb{I}^{\otimes m} = a_0(J) + a_1(J), \ a_i(J) \in V^{\otimes m} \}$$

$$(J^i = \prod_{\mu} \|e_{j_\mu}\|^i, \ i = 0, 1).$$

Now let us take $m = \dim V$, $J = \{1, 2, \dots, m\}$. As we have already seen

$$\langle a_i(J), \bigotimes_{\mu=1}^m e_{\sigma(\mu)} \rangle_{\otimes} = \|a_i(J)\|_{\otimes i} \left\| \bigotimes_{\mu=1}^m e_{\sigma(\mu)} \right\|^{\otimes i}$$

for every permutation σ and every $i = 0, 1$. By the definition of the projective norm in the tensor product space

$$(**) \quad \|a_i(J)\|_{\otimes i} = \min \left\{ \sum_k \prod_j \|x_{kj}\|_i : a_i(J) = \sum_k \bigotimes_j x_{kj}, x_{kj} \in V \right\}.$$

So

$$\begin{aligned} \langle a_i(J), \bigotimes_{\mu=1}^m e_{\sigma(\mu)} \rangle_{\otimes} &= \|a_i(J)\|_{\otimes i} \left\| \bigotimes_{\mu=1}^m e_{\sigma(\mu)} \right\|^{\otimes i} \\ &= \min \left\{ \sum_k \prod_j \|x_{ikj}\|_i : a_i(J) = \sum_k \bigotimes_j x_{ikj}, x_{ikj} \in V \right\} \left\| \bigotimes_{\mu=1}^m e_{\sigma(\mu)} \right\|^{\otimes i} \\ &\geq \sum_k \left\langle \bigotimes_j \tilde{x}_{kj}, \bigotimes_{\mu=1}^m e_{\sigma(\mu)} \right\rangle_{\otimes} = \langle a_i(J), \bigotimes_{\mu=1}^m e_{\sigma(\mu)} \rangle_{\otimes}. \end{aligned}$$

So all inequalities above are really equalities. Therefore for the vectors \tilde{x}_{ikj} , minimizing the expression (**), the following holds:

$$\begin{aligned} \forall \sigma, \prod_j \langle \tilde{x}_{ikj}, e_{\sigma(j)} \rangle &= \left\langle \bigotimes_j \tilde{x}_{ikj}, \bigotimes_{\mu=1}^m e_{\sigma(\mu)} \right\rangle_{\otimes} \\ &= \left\| \bigotimes_j \tilde{x}_{ikj} \right\|_{\otimes i} \left\| \bigotimes_{\mu=1}^m e_{\sigma(\mu)} \right\|^{\otimes i} = \prod_j \|\tilde{x}_{ikj}\|_i \|e_{\sigma(j)}\|^i. \end{aligned}$$

Therefore

$$\langle \tilde{x}_{ikj}, e_{\sigma(j)} \rangle = \|\tilde{x}_{ikj}\|_i \|e_{\sigma(j)}\|^i$$

for every k, j, σ . This means that

$$(***) \quad \langle \tilde{x}_{ikj}, e_p \rangle = \|\tilde{x}_{ikj}\|_i \|e_p\|^i$$

for every i, k, j, p . Since $\mathbb{I}^{\otimes m} = a_0(J) + a_1(J) \neq 0$, then for at least one of $i = 0, 1$ there exists k such that $\tilde{x}_{ikj} \neq 0$ for all j . Then (***) for these i, k, j is equivalent to the assertion that $\tilde{x}_{ikj} / \|\tilde{x}_{ikj}\|_i$ belongs to each of the faces of the unit ball of $\|\cdot\|_i$, whose normal vectors are e_p , $p = 1, \dots, n$. The hyperplanes containing these faces have only one intersection point and this point has to coincide with $\tilde{x}_{ikj} / \|\tilde{x}_{ikj}\|_i$. This fact and the G -invariance of the norm $\|\cdot\|_i$ imply that the unit ball of $\|\cdot\|_i$ is the convex hull of the G -orbit of $\tilde{x}_{ikj} / \|\tilde{x}_{ikj}\|_i$, so the norm $\|\cdot\|_i$ is isometric to the l_{∞}^n -norm. So we have proved the following result:

Theorem 3. *If $\|\cdot\|_{\infty} \in (\dim V, \dim V)\text{-Int}_1(\|\cdot\|_0, \|\cdot\|_1)$ then either both norms $\|\cdot\|_0, \|\cdot\|_1$ are isometric to $\|\cdot\|_{\infty}$, or one of them is proportional to $\|\cdot\|_{\infty}$.*

Concluding remarks. We think that the notion of ω -interpolation spaces is very natural and useful, e.g., for an interpolation theory for nonlinear operators (we are thankful to Yu.A. Brudnyi for this remark). Preliminary investigations of L.Veselova and ourselves show that such a theory seems to be quite interesting, in particular, one can show that it lacks some pathologies usual in interpolation theory for operators. It would be very interesting to construct examples of ω -interpolation functors different from the complex interpolation functor. We hope to return to these questions in other publications.

REFERENCES

1. Yu.A.Brudnyi, S.G.Krein, E.M.Semenov, *Interpolation of linear operators*, in book " Matem. Anal., v.24 (Itogi Nauki i Tekhniki)", (Russian), VINITI AN SSSR, Moscow, 1986, pp. 3–164. MR **88e**:46056
2. L.Veselova, *Duality in Interpolation of Operators*, Ph.D. thesis, Kazan State University, Russia (1991), pp. 1–110.
3. N.Zobin, V.Zobina, *A general theory of sufficient collections of norms with a prescribed semi-group of contractions*, Operator Theory: Advances and Applications **73** (1994), 397–416. MR **96e**:46100
4. V.Zobina, *Interpolation in spaces with prescribed symmetries and uniqueness of sufficient collections*, (Russian), Soobsheniya AN Gruz.SSR **93:2** (1979), 301–303. MR **81b**:46085

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF MIAMI, CORAL GABLES, FLORIDA 33124

E-mail address: zobin@math.miami.edu

Current address: Department of Mathematics, Ohio State University, Columbus, Ohio 43210

E-mail address: zobin@math.ohio-state.edu

DEPARTMENT OF MATHEMATICS, TECHNION – I.I.T., HAIFA, 32000, ISRAEL

Current address: Department of Mathematics and Computer Science, University of Miami, Coral Gables, Florida 33124

E-mail address: zobin@math.miami.edu