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FUNCTIONS OPERATING FROM A COMPLEX BANACH SPACE TO ITS REAL PART

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ABSTRACT. We consider functions operating from a complex Banach function space to its real part. We show among other things, that if $|b| \in \mathrm{Re}B$ for all b in an ultraseparating Banach function space B, then $\mathrm{Re}B = C_{\mathbf{R}}(X)$.

1. INTRODUCTION

Let B be a Banach space of real- or complex-valued continuous functions on a compact Hausdorff space X. A continuous function h defined on an interval of the real line or in a domain of the complex plane is said to *operate* on B if the composition $h \circ b$ belongs to B whenever it is defined, i.e. whenever $b(X)$ is contained in the domain of definition for h.

Quite a number of papers have dealt with operating functions for Banach spaces of continuous functions (e.g. [1], [2], [4], [8], [7], [5] and [6]). In one of these papers [5], the author deals with a related case: Let B be a complex Banach function algebra and h a real-valued function defined in a domain in the complex plane such that $h \circ b \in \mathrm{Re}B$, the space of real parts of functions in B, whenever the composition of h with b is defined. What can be said about B or h? More generally one can replace complex Banach function algebras by complex Banach function spaces. Operating functions for real Banach spaces is a special case because if h operates on a real Banach function space B, then the function $h_1(s+it) = h(s)$ operates from $B_1 = B + iB$ to $\mathrm{Re}B_1 = B$.

In [5] O. Hatori shows that if B is a function algebra and $B \neq C(X)$, then only harmonic functions can operate from B to $\mathrm{Re}B$. For general Banach function algebras, such as the algebra of continuously differentiable complex-valued functions on the unit interval with the norm given by $\|f\| = \|f\|_\infty + \|f'\|_\infty$, the class of functions operating from the algebra to its real part is much larger.

In the process of characterizing operating functions for a function algebra one shows that if a non-harmonic function operates, then the function algebra must be a Dirichlet algebra. Now, a Dirichlet algebra on a compact Hausdorff space X is ultraseparating on X (definition below) and this property, first studied by A. Bernard [1], is crucial in the proofs. For ultraseparating Banach function algebras the result is the same as for function algebras (see [5]).

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In this note we study the more general case when B is a Banach function space. We show that there exists an ultraseparating Banach function space B, with $\mathrm{Re}B \neq C_{\mathbf{R}}(X)$, such that the function $h(z) = |z|^2$ operates from B to $\mathrm{Re}B$. Otherwise the results are similar to the ones in the algebra case, some a bit weaker though.

We prove that if B is an ultraseparating Banach function space on X, for which there is a continuous function h operating from B to $\mathrm{Re}B$ which is neither harmonic nor of the type $h(z) = |z|^2$ on any open set, then $\mathrm{Re}B = C_{\mathbf{R}}(X)$. Thus if $h(z) = |z|^p$, where $p \neq 2$, operates from B to $\mathrm{Re}B$, then $\mathrm{Re}B = C_{\mathbf{R}}(X)$. In particular, if the function $h(z) = |z|$ operates from B to $\mathrm{Re}B$, then $\mathrm{Re}B = C_{\mathbf{R}}(X)$. This contrasts with the real case where there is an example of an ultraseparating Banach function space which is not all of $C_{\mathbf{R}}(X)$, but for which the function $h(t) = |t|$ operates. (See [5].) We also give an example of a Banach function space, which is not all of $C(X)$, whose real part equals $C_{\mathbf{R}}(X)$.

2. The main results

In what follows, unless otherwise stated, B will denote a complex Banach function space on a compact Hausdorff space X. This means that B separates the points of X, contains the constant functions, and that the norm $\|\cdot\|$ on B dominates the sup-norm $\|\cdot\|_\infty$. Then $\mathrm{Re}B$, the space of real parts of functions in B, is a real Banach function space on X in the norm $\|u\| = \inf\{\|b\| : b \in B \text{ and } \mathrm{Re}b = u\}$. (We use the same symbol for the norm on $\mathrm{Re}B$ as for the norm on B.)

When trying to show that only functions of a special kind can operate on a Banach function space or, from a Banach function space to its real part, one tries to show that the Banach function space in question equals $C(X)$, if it has an operating function not of this special kind. The main tool is *Bernard's Lemma*. The space $l^\infty(B)$ of all bounded sequences of functions from B is a subspace of $l^\infty(C(X))$, the space of all bounded sequences of continuous functions on X. Bernard's Lemma says that if $l^\infty(B)$ is dense in $l^\infty(C(X))$, then $B = C(X)$.

Now, the space $l^\infty(C(X))$ can be realized in a natural way as the space of all continuous functions on a compact Hausdorff space, namely the space $\beta(\mathbf{N}\times X)$, the Stone-Čech compactification of $\mathbf{N}\times X$. The correspondance is given by $\{f_n\}(k,x) = f_k(x)$ for $\{f_n\} \in l^\infty(C(X))$ and $(k,x) \in \mathbf{N}\times X$. Thus, a necessary condition for $l^\infty(B)$ to be dense in $l^\infty(C(X))$, is that $l^\infty(B)$ separates the points of $\beta(\mathbf{N}\times X)$. When that condition is satisfied we say that B is *ultraseparating* on X. If e.g. B is a Dirichlet algebra on X, then B and $\mathrm{Re}B$ are ultraseparating on X, (see [1]). There are also intrinsic characterizations of the ultraseparation property. We will state and make use of one of those later on.

We begin with a result for subspaces, not necessarily ultraseparating, of $C_{\mathbf{R}}(X)$.

Theorem 2.1. *Let B be a subspace of $C(X)$ containing the constant functions and let h be a continuous real-valued function defined on the open unit disc U of the complex plane, such that $h\circ b \in \overline{\mathrm{Re}B}$, the closure of $\mathrm{Re}B$ in $C(X)$, for all $b \in B$ for which $b(X) \subset U$. If h is neither harmonic nor of the type $h(s+it) = \alpha(s^2+t^2)+\beta s+\gamma t+\delta$, then $\mathrm{Re}B$ is dense in $C_{\mathbf{R}}(X)$.*

Proof. Let us first suppose that h is a C^∞-function in $U_\delta = \{z \in \mathbf{C} : |z| < \delta\}$ where $0 < \delta < 1$. Let $u, v \in B$ with $\|u\|_\infty < \delta$ and $\|v\|_\infty < 1$ and define for each $\xi \in \mathbf{R}$ with $|\xi| < \delta - \|u\|_\infty$ an element $k(\xi)$ of $\overline{\mathrm{Re}B}$ by putting $k(\xi) = h\circ(u+\xi v)$.

Differentiating twice w.r.t. ξ and then putting $\xi = 0$ we deduce that

$$(h_{ss} \circ u)v_1^2 + 2(h_{st} \circ u)v_1 v_2 + (h_{tt} \circ u)v_2^2 \in \overline{\mathrm{Re}B} \tag{1}$$

where the indices on h denote partial derivatives and where v_1 and v_2 are the real and imaginary parts of v. Replacing v by iv and by $v + iv$ we find that

$$(h_{ss} \circ u)v_2^2 - 2(h_{st} \circ u)v_1 v_2 + (h_{tt} \circ u)v_1^2 \in \overline{\mathrm{Re}B} \tag{2}$$

and that

$$(h_{ss} \circ u)(v_1 - v_2)^2 + 2(h_{st} \circ u)(v_1^2 - v_2^2) + (h_{tt} \circ u)(v_1 + v_2)^2 \in \overline{\mathrm{Re}B}. \tag{3}$$

Adding (1) and (2) we deduce that

$$(\Delta h \circ u)(v_1^2 + v_2^2) \in \overline{\mathrm{Re}B} \tag{4}$$

and subtracting (2) from (1) that

$$(h_{ss} \circ u - h_{tt} \circ u)(v_1^2 - v_2^2) + 4(h_{st} \circ u)s v_1 v_2 \in \overline{\mathrm{Re}B}. \tag{5}$$

Also, subtracting (1) and (2) from (3) gives

$$(h_{tt} \circ u - h_{ss} \circ u)v_1 v_2 + (h_{st} \circ u)(v_1^2 - v_2^2) \in \overline{\mathrm{Re}B}. \tag{6}$$

Now, let u be a constant function. Multiplying (5) with $h_{ss} \circ u - h_{tt} \circ u$ and (6) with $4(h_{st} \circ u)$ and adding, we find that

$$((h_{ss} \circ u - h_{tt} \circ u)^2 + 4(h_{st} \circ u)^2)(v_1^2 - v_2^2) \in \overline{\mathrm{Re}B}. \tag{7}$$

If $u(x) = z$ for all $x \in X$, (4) and (7) show that

$$\Delta h(z)(v_1^2 + v_2^2) \in \overline{\mathrm{Re}B} \tag{8}$$

and that

$$((h_{ss}(z) - h_{tt}(z))^2 + 4(h_{st}(z))^2)(v_1^2 - v_2^2) \in \overline{\mathrm{Re}B} \tag{9}$$

for all $z \in U_\delta$ and all $v = v_1 + iv_2 \in B$.

We are assuming that h is not harmonic in U. Replacing h by $h_1(z) = h(\alpha z + z_0)$ for a suitable choice of α and z_0 we can from the start assume that h is not harmonic in any neighbourhood of 0. Thus, there is a $z_1 \in U_\delta$ for which $\Delta h(z_1) \neq 0$. Also, using the other assumptions on h, we may assume that there is a $z_2 \in U_\delta$ for which $(h_{ss}(z_2) - h_{tt}(z_2))^2 + 4(h_{st}(z_2))^2 \neq 0$. From (8) and (9) it then follows that $v_1^2 \in \overline{\mathrm{Re}B}$ for all $v_1 \in \mathrm{Re}B$. The Stone-Weierstrass Theorem now implies that $\mathrm{Re}B$ is dense in $C_{\mathbf{R}}(X)$.

For the general case we first note that if $\varphi \in C_0^\infty(\mathbf{C})$ has support in U_δ, then $(h * \varphi)(z)$ is defined for $z \in U_{1-\delta}$. Approximating the integral defining $h * \varphi$ with Riemann sums we deduce that $(h * \varphi) \circ v \in \overline{\mathrm{Re}B}$ if $v(X) \subset U_{1-\delta}$.

If h is not a C_0^∞-function in any neighbourhood of 0, we can find a C_0^∞-function φ such that $h * \varphi$ is not harmonic in a neighbourhood of 0. From the first part of the proof it follows that $v_1^2 + v_2^2 \in \overline{\mathrm{Re}B}$ for all $v = v_1 + iv_2 \in B$. We can also find a C_0^∞-function ψ such that $h * \psi$ is not a polynomial of degree 2. From the first part of the proof it follows that $v_1^2 - v_2^2 \in \overline{\mathrm{Re}B}$ for all $v = v_1 + iv_2 \in B$. We conclude that $v_1^2 \in \overline{\mathrm{Re}B}$ for all $v_1 \in \mathrm{Re}B$. As in the first part it follows that $\mathrm{Re}B$ is dense in $C_{\mathbf{R}}(X)$. □

The next examples show why the restrictions on h in Theorem 2.1 are necessary.

Example 2.2. Let A be a function algebra on a compact Hausdorff space X and let h be a harmonic function defined in the open unit disc U. Then h operates from A to ReA. To see this let a be a function in A for which $a(X) \subset U$. Since $a(X)$ is a compact subset of U, we can find a smaller open disc U_δ containing $a(X)$ and a harmonic function k on U such that the restriction of $h + ik$ to U_δ is a bounded analytic function. Then the function $(h + ik) \circ a$ belongs to A and its real part is $h \circ a$.

Example 2.3. Let B be the Banach function space on the unit circle Γ of the complex plane spanned by the functions $b(z) = \lambda + \mu z$ where λ and μ are complex numbers, and let $h(s + it) = s^2 + t^2$. Then

$$(h \circ b)(s + it) = |\lambda|^2 + |\mu|^2 + 2(\lambda_1\mu_1 + \lambda_2\mu_2)s + 2(\lambda_2\mu_1 - \lambda_1\mu_2)t$$

where $\lambda = \lambda_1 + i\lambda_2$ and where $\mu = \mu_1 + i\mu_2$, which shows that $h \circ b \in \mathrm{Re}B$.

For the next result we impose slightly stronger conditions on the operating function h. Before stating that result we need the following characterization of the ultraseparation property:

Proposition 2.4. *Let B be an ultraseparating Banach function space on a compact Hausdorff space X. Then there exists a natural number N and a positive number M, such that for any $f \in C_{\mathbf{R}}(X)$ with $\|f\|_\infty \leq 1$, one can find functions $b_{i,k_i} \in B_M$, the M-ball of B, where $1 \leq i, k_i \leq N$, and numbers $\varepsilon_i = \pm 1$, such that*

$$\|f - \sum_{i=1}^{N} \varepsilon_i |b_{i1}|^2 |b_{i2}|^2 \cdots |b_{ik_i}|^2\|_\infty < 1/2.$$

Proof. If the statement of the proposition is false, then we can find an increasing unbounded sequence $\{M_n\}$ of positive numbers such that there exists for each pair n, M_n a function $f_n \in C_{\mathbf{R}}(X)$ with $\|f_n\|_\infty \leq 1$, such that the condition is not satisfied for n or M_n.

Let $l^\infty(|B|^2)$ denote the linear span of all the sequences $\{|b_n|^2\}$, where $\{b_n\} \in l^\infty(B)$. Then $l^\infty(|B|^2)$ separates the points of $\beta(\mathbf{N} \times X)$, and thus there is an element $\{g_n\} \in \mathrm{alg}(l^\infty(|B|^2))$, the algebra generated by $l^\infty(|B|^2)$, such that $\|\{f_n\} - \{g_n\}\|_\infty < 1/2$. Since $\{g_n\}$ is a finite sum of products of elements from $l^\infty(|B|^2)$, we have reached a contradiction. □

Corollary 2.5. *If B is ultraseparating on X, then there is a natural number m such that $l^\infty(|B|^2)^m$ is dense in $l^\infty(C_{\mathbf{R}}(X))$, where $l^\infty(|B|^2)^m$ denotes the linear span of all products of at most m elements of $l^\infty(|B|^2)$.*

Theorem 2.6. *Let B be an ultraseparating Banach function space on a compact Hausdorff space X and let h be a continuous function, defined on an open subset O of the complex plane, which operates from B to* ReB. *Suppose h is neither harmonic nor of the type $h(s + it) = \alpha(s^2 + t^2) + \beta s + \gamma t + \delta$ on any open subset of O. Then* Re$B = C_{\mathbf{R}}(X)$.

Proof. Pick a function b_1 in B and a positive number r such that $(b_1 + b)(X) \subset O$ if b is in the r-ball B_r of B. Now,

$$b_1 + B_r = \bigcup_n \{b \in B_r : \|h \circ (b_1 + b)\| \leq n\}.$$

By the Baire Category Theorem there are a function $b_0 \in B$ and positive numbers ϵ and M such that

$$\|h \circ (b_0 + b)\| \le M$$

for all b in some dense subset of the ϵ-ball B_ϵ of B. Hence

$$h \circ \{b_0 + b_n\} \in \overline{l^\infty(\mathrm{Re}B)}, \tag{10}$$

the closure of $l^\infty(\mathrm{Re}B)$ in $C(\beta(\mathbf{N} \times X))$, if $b_n \in B_\epsilon$ for all n.

The proof now splits into two parts, depending on whether h is a C^∞-function or not.

Part 1. Let V be an open subset of O such that h is a C^∞-function on V, and let $\lambda \in V$ be chosen such that $\Delta h(\lambda) \neq 0$. We let $b_1 \equiv \lambda$ and choose a smaller r if necessary to ensure that $\lambda + b(X) \subset V$ for all b in B_r.

We now proceed as in the proof of Theorem 2.1. Differentiating $h \circ \{b_0 + \xi b_n\}$ we deduce that

$$(\{u_n\}^2 + \{v_n\}^2)(\Delta h \circ b_0) \in \overline{l^\infty(\mathrm{Re}B)} \tag{11}$$

for all $\{b_n\} = \{u_n\} + i\{v_n\} \in l^\infty(B)$.

If r tends to 0, then b_0 tends to b_1 so that by (11),

$$(\{u_n\}^2 + \{v_n\}^2)\Delta h(\lambda) \in \overline{l^\infty(\mathrm{Re}B)},$$

and hence

$$\{u_n\}^2 + \{v_n\}^2 \in \overline{l^\infty(\mathrm{Re}B)} \tag{12}$$

for all $\{b_n\} = \{u_n\} + i\{v_n\} \in l^\infty(B)$.

The conditions on h imply that there is another $\lambda \in V$ for which

$$(h_{ss}(\lambda) - h_{tt}(\lambda))^2 + 4(h_{st}(\lambda))^2 \neq 0.$$

We differentiate $h \circ \{b_0 + \xi b_n\}$, proceed as in the proof of Theorem 2.1 and then let b_0 tend to b_1, and find that

$$(h_{ss}(\lambda) - h_{tt}(\lambda))(\{u_n\}^2 - \{v_n\}^2) + 4h_{st}(\lambda)\{u_n\}\{v_n\} \in \overline{l^\infty(\mathrm{Re}B)} \tag{13}$$

and that

$$(h_{tt}(\lambda) - h_{ss}(\lambda))\{u_n\}\{v_n\} + h_{st}(\lambda)(\{u_n\}^2 - \{v_n\}^2) \in \overline{l^\infty(\mathrm{Re}B)} \tag{14}$$

for all $\{b_n\} = \{u_n\} + i\{v_n\} \in l^\infty(B)$.

From (13) and (14) we deduce that

$$\{u_n\}^2 - \{v_n\}^2 \in \overline{l^\infty(\mathrm{Re}B)} \tag{15}$$

for all $\{b_n\} = \{u_n\} + i\{v_n\} \in l^\infty(B)$.

It follows from (12) and (15) that $\{\mathrm{Re}b_n\}^2 \in \overline{l^\infty(\mathrm{Re}B)}$ for all $\{b_n\} \in l^\infty(B)$. We apply the Stone-Weierstrass Theorem to deduce that $l^\infty(\mathrm{Re}B)$ is dense in $l^\infty(C_{\mathbf{R}}(X))$. Bernard's Lemma now implies that $\mathrm{Re}B = C_{\mathbf{R}}(X)$.

Part 2. Now we are supposing that h is not a C^∞-function on any open subset of O. If φ is a C_0^∞-function on $\mathbf{C}$ with support in a disc with center 0 and radius $\varepsilon/2$, then it follows from (10) that

$$h_\varphi \circ \{b_0 + b_n\} \in \overline{l^\infty(\mathrm{Re}B)}$$

if $b_n \in B_{\varepsilon/2}$ for all n, where $h_\varphi = h * \varphi$.

Let $x_0 \in X$ and put $\lambda = b_0(x_0)$. Since h is not a C^∞-function in any neighbourhood of λ, we can find φ as above and $\lambda_1 \in \mathbf{C}$ with $|\lambda - \lambda_1| < \varepsilon/2$ such that $\Delta^m h_\varphi(\lambda_1) \neq 0$, where m is chosen using Proposition 2.4 so that $l^\infty(|B|^2)^m$ is dense in $l^\infty(C_{\mathbf{R}}(X))$.

Differentiating $h_\varphi \circ \{b_0 + (\lambda_1 - \lambda) + \xi b_n + \eta v_n\}$ repeatedly we deduce that

$$|b_{1n}|^2 \cdots |b_{mn}|^2 \Delta^m h_\varphi \circ \{b_0 + \lambda_1 - \lambda\} \in \overline{l^\infty(\mathrm{Re} B)}$$

for all $\{b_{1n}\}, \ldots, \{b_{mn}\} \in l^\infty(B)$ and hence

$$l^\infty(C_{\mathbf{R}}(X)) \Delta^m h_\varphi \circ \{b_0 + \lambda_1 - \lambda\} \subseteq \overline{l^\infty(\mathrm{Re} B)}.$$

Let $f = \Delta^m h_\varphi \circ (b_0 + \lambda_1 - \lambda)$. The inclusion above shows that

$$l^\infty(C_{\mathbf{R}}(X))\{f\}^2 \subseteq l^\infty(C_{\mathbf{R}}(X))\{f\} \subseteq \overline{l^\infty(\mathrm{Re} B)}.$$

Since $\lambda = b_0(x_0)$, it follows that $f(x_0) = \Delta^m h_\varphi(\lambda_1) \neq 0$ and hence $f(x) \neq 0$ for x in a neighbourhood of x_0. Since X is compact, we can choose finitely many such functions f such that if g is the sum of the squares of these functions, then g is strictly positive on X. From the above inclusion we deduce that

$$l^\infty(C_{\mathbf{R}}(X))\{g\} \subseteq \overline{l^\infty(\mathrm{Re} B)}.$$

Since $\{g\} > 0$ on $\beta(\mathbf{N} \times X)$, it follows that $l^\infty(\mathrm{Re} B)$ is dense in $l^\infty(C_{\mathbf{R}}(X))$ and thus by Bernard's Lemma, $\mathrm{Re} B = C_{\mathbf{R}}(X)$.

□

Corollary 2.7. *If h, defined in an open subset O of the complex plane, operates from B to ReB and if Δh is not constant on any open subset of O, in particular if h is not differentiable on any open subset of O, then ReB $= C_{\mathbf{R}}(X)$.*

As an example of functions satisfying the conditions of the corollary above, and hence of Theorem 2.6, we mention the functions $h(z) = |z|^p$, where $p > 0$ and $p \neq 2$. In particular $p = 1$ gives the following result:

Corollary 2.8. *If $h(z) = |z|$ operates from B to* ReB*, then* $\mathrm{Re} B = C_{\mathbf{R}}(X)$.

As we said in the introduction, there are Banach function spaces $B \neq C(X)$ for which $\mathrm{Re} B = C_{\mathbf{R}}(X)$.

Example 2.9. Let $X = [0,1] \cup [2,3]$ and let $\tau : X \to X$ be the homomorphism

$$\tau(x) = \begin{cases} 2 + t & \text{if } t \in [0,1], \\ t - 2 & \text{if } t \in [2,3]. \end{cases}$$

Let $T : C_{\mathbf{R}}(X) \to C_{\mathbf{R}}(X)$ be the map

$$(Tf)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in [0,1], \\ -f(\tau(t)) & \text{if } t \in [2,3]. \end{cases}$$

Now put

$$B = \{f + iTf + \lambda : f \in C_{\mathbf{R}}(X) \text{ and } \lambda \in C\}.$$

Since $T^2 = -I$, the space B is a complex subspace of $C(X)$. It is clear that B is uniformly closed.

We can ask what happens if we impose on h the weaker conditions from Theorem 2.1. Then we can no longer conclude that $\mathrm{Re}B = C_{\mathbf{R}}(X)$. As mentioned earlier, there is an example by O. Hatori of an ultraseparating real Banach function space $B \neq C_{\mathbf{R}}(X)$ such that the function $h(t) = |t|$ operates from B to B. If we put $B_1 = B + iB$ and $h(s+it) = |s|$, then h satisfies the conditions from Theorem 2.1 and operates from B_1 to $\mathrm{Re}B_1$, but $\mathrm{Re}B_1 = B \neq C_{\mathbf{R}}(X)$.

The best result we have obtained for the case when h satisfies the conditions of Theorem 2.1 is that, except for finitely many points of X, $\mathrm{Re}B$ is locally a $C_{\mathbf{R}}(K)$-space, meaning that every x not in some finite subset of X has a compact neighbourhood K_x for which $\mathrm{Re}B|K_x = C_{\mathbf{R}}(K_x)$. This result corresponds to a similar result for operating functions on an ultraseparating real Banach function space (see [3]). One might ask whether in this case the stronger conclusion holds, that $\mathrm{Re}B$ contains every continuous function vanishing in a neighbourhood of some finite subset of X. This result, if true, would also correspond to a similar result for real Banach function spaces (see [4]).

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