PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 3, March 1997, Pages 709–716
S 0002-9939(97)03670-8

SERRE-DUALITY FOR TAILS(A)

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(Communicated by Ken Goodearl)

ABSTRACT. A version of Serre-duality is proved for Artin's non-commutative projective schemes.

0. INTRODUCTION

Let k be a field and A a left noetherian $\mathbb{N}$-graded k-algebra. If A is commutative, it describes a geometric object, the projective k-scheme $\operatorname{Proj}(A)$. Most of the geometry on $\operatorname{Proj}(A)$ really is about sheaves of coherent or quasi-coherent modules on $\operatorname{Proj}(A)$; thus, the abelian categories of such sheaves play prominent roles in algebraic geometry.

After dropping the condition that A should be commutative, it is possible to define abelian categories generalizing the categories of (quasi-)coherent sheaves. These categories, $\operatorname{tails}(A)$ and $\operatorname{Tails}(A)$, are obtained as the quotients

$$\begin{aligned} \operatorname{tails}(A) &= \operatorname{grmod}(A)/\operatorname{tors}(A), \\ \operatorname{Tails}(A) &= \operatorname{GrMod}(A)/\operatorname{Tors}(A); \end{aligned}$$

here $\operatorname{grmod}(A)$ is the category of finitely generated graded left A-modules and graded homomorphisms, and $\operatorname{tors}(A)$ is the dense subcategory of torsion-modules; $\operatorname{GrMod}(A)$ and $\operatorname{Tors}(A)$ are the corresponding objects without the finiteness condition. $\operatorname{tails}(A)$ is a full subcategory of $\operatorname{Tails}(A)$. Details of the construction can be found in [1], and generalities on quotient-categories in [3].

Using the philosophy that $\operatorname{tails}(A)$ and $\operatorname{Tails}(A)$ generalize notions from (commutative) geometry, it is natural to ask if they support generalizations of various well-known geometrical constructions. In particular, it is natural to look for a duality-theory corresponding to Serre-duality, which is one of the most important technical tools in modern commutative geometry.

Depending on how "good" the k-scheme X is, classical Serre-duality yields functorial isomorphisms

$$H^n(X,\mathcal{F})' \cong \operatorname{Ext}_X^{d-n}(\mathcal{F},\omega^\circ)$$

for a range of n-values. Here d is the dimension of X, and $\mathcal{F}$ is a quasi-coherent sheaf on X. The prime denotes dualization with respect to k, and Ext_X is the usual Ext-functor on the abelian category of quasi-coherent sheaves, while H^n is the n'th

Received by the editors March 9, 1995 and, in revised form, September 25, 1995.
1991 *Mathematics Subject Classification.* Primary 14A22, 16W50; Secondary 18E30.
Key words and phrases. Non-commutative projective schemes, Serre-duality, Brown Adjoint Functor Theorem.

right derived functor of the global section-functor, Γ. The sheaf ω° is a "dualizing object".

In more advanced versions of the theory it is possible, by replacing ω° by a complex of sheaves ("the dualizing complex"), to prove the duality-formula above with fewer assumptions; one then has to use a hyper-Ext instead of the usual Ext.

In their paper [7], A. Yekutieli and J. Zhang construct a theory of Serre-duality for tails(A). This paper does the same thing but on Tails(A) and with fewer conditions on A; also, the theory presented here is hypercohomological, yielding Serre-duality, corollary 3.5, in the version with a dualizing complex as mentioned above. Classical Serre-duality, corollary 3.6, is obtained as a special case. The paper follows hypercohomological ideas from the commutative theory presented in [6], and uses the Brown Adjoint Functor Theorem, [6, thm. 4.1, p. 223]. The main results are corollary 3.3 and its special cases, corollaries 3.5 and 3.6.

A bit about the notation: throughout the paper, k is a field and A is a left-noetherian $\mathbb{N}$-graded k-algebra. We have the canonical projection-functor,

$$\pi : \mathrm{GrMod}(A) \longrightarrow \mathrm{Tails}(A),$$

and its right-adjoint,

$$\omega : \mathrm{Tails}(A) \longrightarrow \mathrm{GrMod}(A).$$

$\pi\omega$ is equivalent to the identity-functor, cf. [3, III.2, prop. 3, p. 371]. Objects in Tails(A) are denoted by calligraphic letters, and $\mathcal{A} = \pi A$; twisting in Tails(A) is denoted by ( ). We call

$$\Gamma(-) = \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{A}, -) : \mathrm{Tails}(A) \longrightarrow \mathrm{Vect}(k)$$

the section-functor. The n'th right-derived functor of Γ is denoted H^n, and the cohomological dimension of Γ is denoted $d = \mathrm{cd}(\mathrm{Tails}(A))$.

An isomorphism in any of the categories that come into play will be symbolized by "$\xrightarrow{\cong}$". If $\mathfrak{A}$ is an abelian category, complexes of $\mathfrak{A}$-objects are equipped with a "$*$" as superscript, and Hom^* is complex-Hom. $K(\mathfrak{A})$ is the category of complexes of $\mathfrak{A}$-objects and homotopy-classes of chain-maps, while $D(\mathfrak{A})$ is the derived category of $\mathfrak{A}$ (these are categories of unbounded complexes). The symbol h^n will denote the n'th cohomology-functor on these categories; H^n is reserved for the n'th derived functor of Γ. A quasi-isomorphism from one complex to another is symbolized by "$\xrightarrow{\simeq}$", and twisting of complexes is denoted by [ ]. If $T : K(\mathfrak{A}) \longrightarrow K(\mathfrak{B})$ is a triangulated functor, its derived functor is denoted $\underline{\underline{R}}T : D(\mathfrak{A}) \longrightarrow D(\mathfrak{B})$ (but may not exist!) We write $D(\mathcal{A})$ resp. $D(k)$ for the derived categories $D(\mathrm{Tails}(A))$ resp. $D(\mathrm{Vect}(k))$.

Acknowledgement. I would like to thank Anders Jensen for introducing me to non-commutative geometry and for discussions and encouragement, and Paul Smith and James Zhang for sharing their ideas on the subject with me. I also thank my teacher, Anders Thorup, for all sorts of help, especially with the proof of lemma 2.3.

1. Some facts about Tails(A)

This section proves a number of easy but useful properties, mostly of a homological nature, about Tails(A).

Proposition 1.1. *The category* Tails(A) *has all small direct limits.*

Proof. Since Tails(A) is abelian it has all cokernels and thus all coequalizers. Now π is a left-adjoint, so it preserves direct limits. In particular, it preserves small direct sums, and since any object in Tails(A) has the form πM for some $M \in \mathrm{GrMod}(A)$, cf. [3, III.1, p. 365], the functor π provides Tails(A) with small direct sums. But a category with coequalizers and small direct sums has all small direct limits, cf. [5, V.2, cor. 2, p. 109]. □

Lemma 1.2. *In* Tails(A) *small direct sums preserve exactness.*

Proof. Given a small system of exact sequences in Tails(A),

$$0 \to \mathcal{M}_i' \longrightarrow \mathcal{M}_i \longrightarrow \mathcal{M}_i'' \to 0.$$

According to [3, III.1, cor. 1, p. 368], each sequence can be obtained as

$$0 \to \pi M_i' \xrightarrow{\pi(\mu_i')} \pi M_i \xrightarrow{\pi(\mu_i)} \pi M_i'' \to 0,$$

where the sequences of M_i's are short-exact in GrMod(A). Now the sequence

$$0 \to \bigoplus_i M_i' \longrightarrow \bigoplus_i M_i \longrightarrow \bigoplus_i M_i'' \to 0$$

is clearly short-exact, but applying the exact functor π yields the direct sum of the $\mathcal{M}_i$-sequences, so this sum is exact. □

Corollary 1.3. *Taking (co-)homology of* Tails(A)*-complexes commutes with taking small direct sums of such complexes.*

Lemma 1.4. *In* Tails(A) *the direct sum of a small family of injective objects is again an injective object.*

Proof. A is left-noetherian, so the corresponding statement for GrMod(A) is true. Now let $\{\mathcal{E}_i\}_{i \in I}$ be a small family of injectives from Tails(A). Then for each i we know that $\mathcal{E}_i = \pi\omega\mathcal{E}_i$ and that $\omega\mathcal{E}_i$ is a graded-injective and torsionfree graded module, cf. [1, prop. 7.1(1), p. 270]. The sum $\bigoplus_i \omega\mathcal{E}_i$ is thus also graded-injective and torsionfree, and applying π yields an injective object of Tails(A), cf. [1, prop. 7.1(3), p. 270] — but

$$\pi \bigoplus_i \omega\mathcal{E}_i \cong \bigoplus_i \pi\omega\mathcal{E}_i \cong \bigoplus_i \mathcal{E}_i.$$

□

Corollary 1.5. *Let* $\mathfrak{A}$ *be an abelian category and let* $T : \mathrm{Tails}(A) \longrightarrow \mathfrak{A}$ *be an additive functor which commutes with small direct sums. Then each right-derived functor* R^iT *also commutes with small direct sums.*

Proof. An immediate consequence of lemma 1.4 and corollary 1.3. □

Corollary 1.6. *If* $\mathcal{M} \in \mathrm{tails}(A)$*, then each functor*

$$\mathrm{Ext}^i_{\mathrm{Tails}(A)}(\mathcal{M}, -) : \mathrm{Tails}(A) \longrightarrow \mathrm{Vect}(k)$$

commutes with small direct sums.

Proof. Because of corollary 1.5 it is enough to see that $\mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{M}, -)$ commutes with small direct sums, but this is a simple consequence of the fact that if M is in grmod(A), then the functor

$$\mathrm{Hom}_{\mathrm{GrMod}(A)}(M, -) : \mathrm{GrMod}(A) \longrightarrow \mathrm{Vect}(k)$$

commutes with small direct sums. □

2. Hypercohomology for Tails(A)

Here we study the hypercohomology of Tails(A). Recall that d denotes the cohomological dimension of the functor $\Gamma(-) = \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{A}, -)$. All the results below involving Γ presuppose $d < \infty$.

Proposition 2.1. *The categories $D(\mathcal{A})$ and $D(k)$ have small direct sums, and if $T(-) = \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{A}(m), -)$ and $d < \infty$ then the derived functor*

$$\underline{\underline{R}}T : D(\mathcal{A}) \longrightarrow D(k)$$

exists and preserves small direct sums. In particular this is true for $\underline{\underline{R}}\Gamma$.

Proof. Small direct sums in the categories in question are obtained as the ordinary direct sums of the relevant complexes, cf. [2, sect. 1, pp. 211-213].

Twisting by a fixed amount is an auto-equivalence of the category Tails(A), cf. [1, p. 236], so for $\mathcal{M}, \mathcal{N} \in \mathrm{Tails}(A)$ and $m \in \mathbb{Z}$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{M}(m), \mathcal{N}) &\cong \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{M}(m)(-m), \mathcal{N}(-m)) \\ &\cong \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{M}, \mathcal{N}(-m)). \end{aligned}$$

If S denotes twisting by $-m$ we see that $T \cong \Gamma \circ S$, and S is exact and preserves injectives, so

$$R^n T \cong H^n \circ S.$$

Since Γ has cohomological dimension d, so does T, i.e. the derived functors $R^n T$ vanish precisely from number $d+1$ onwards. Since Tails(A) has enough injectives, this is sufficient by [4, cor. I.5.3(γ), p. 56] to secure the existence of $\underline{\underline{R}}T$.

Finally, suppose that $\mathcal{C}^* \in D(\mathcal{A})$ has the resolution $\mathcal{C}^* \xrightarrow{\simeq} \mathcal{Q}^*$, meaning that the arrow is a quasi-isomorphism. Suppose also that $\mathcal{Q}^*$ consists of T-acyclic objects; such a resolution exists by [4, proof of cor. I.5.3, p. 58]. Then

$$\underline{\underline{R}}T(\mathcal{C}^*) = T(\mathcal{Q}^*),$$

cf. [4, proof of thm. I.5.1, p. 54]. Because of corollary 1.3, if we have resolutions $\mathcal{C}_i^* \xrightarrow{\simeq} \mathcal{Q}_i^*$ of a small family of complexes, $\bigoplus_i \mathcal{C}_i^* \xrightarrow{\simeq} \bigoplus_i \mathcal{Q}_i^*$ is also a resolution, and because of corollary 1.6, if all the $\mathcal{Q}_i^*$'s consist of T-acyclics, so will $\bigoplus_i \mathcal{Q}_i^*$. But then

$$\underline{\underline{R}}T(\bigoplus_i \mathcal{C}_i^*) = T(\bigoplus_i \mathcal{Q}_i^*) \cong \bigoplus_i T(\mathcal{Q}_i^*) = \bigoplus_i \underline{\underline{R}}T(\mathcal{C}_i^*).$$

□

Lemma 2.2. *Let $\mathcal{C}^* \in D(\mathcal{A})$ have $h^n \mathcal{C}^* \neq 0$. Then for any $M \in \mathbb{N}$ there exists an $m \geq M$ such that*

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{A}(-m)[-n], \mathcal{C}^*) \neq 0.$$

Proof. $\mathcal{C}^*$ is a complex over Tails(A), so $C^* = \omega \mathcal{C}^*$ is a complex over GrMod(A) and $\mathcal{C}^* \cong \pi\omega\mathcal{C}^* = \pi C^*$. Since π is exact and $h^n \mathcal{C}^* \neq 0$, we see that $h^n C^*$ cannot be torsion. So $h^n C^*$ has homogeneous non-torsion elements of degree larger than M for any $M \in \mathbb{N}$. We can thus construct

$$A(-m) \longrightarrow Z^n C^* \longrightarrow\!\!\!\!\!\rightarrow h^n C^*$$

with non-torsion image in $h^n C^*$ for some $m \geq M$, yielding a chain-map

$$A(-m)[-n] \longrightarrow C^*$$

whose induced map in cohomology has non-torsion image. But then $\mathcal{A}(-m)[-n] \longrightarrow \mathcal{C}^*$ does not induce the zero-map on cohomology, whence

$$\operatorname{Hom}_{D(\mathcal{A})}(\mathcal{A}(-m)[-n], \mathcal{C}^*) \neq 0.$$

□

Lemma 2.3. *Let $\mathfrak{A}$ be an abelian k-linear category with enough injectives, let $A \in \mathfrak{A}$, put $T(-) = \operatorname{Hom}_{\mathfrak{A}}(A, -)$ and suppose that $R^{N+1}T = R^{N+2}T = \cdots = 0$ for some $N \in \mathbb{N}$. Then the derived functor*

$$\underline{\underline{R}}T : D(\mathfrak{A}) \longrightarrow D(k)$$

exists and satisfies

$$h^0 \underline{\underline{R}}T(-) \xrightarrow{\cong} \operatorname{Hom}_{D(\mathfrak{A})}(A, -).$$

Proof. Existence of $\underline{\underline{R}}T$ follows from [4, cor. I.5.3(γ), p. 56].

The isomorphism will be constructed as follows: any $\mathfrak{A}$-complex is quasi-isomorphic to a complex of T-acyclic objects, so it will be sufficient to come up with isomorphisms

$$\varphi : h^0 \underline{\underline{R}}T(Q^*) \xrightarrow{\cong} \operatorname{Hom}_{D(\mathfrak{A})}(A, Q^*)$$

for Q^*'s consisting of T-acyclics. Now, $h^0 \underline{\underline{R}}T(Q^*) = h^0 \operatorname{Hom}_{\mathfrak{A}}(A, Q^*)$ is the k-space of homotopy-classes of chain-maps from A to Q^*. We may thus define φ by letting it send a class represented by $a : A \longrightarrow Q^*$ to the $D(\mathfrak{A})$-morphism represented by the diagram

$$\begin{array}{ccccc} & & Q^* & & \\ & \overset{a}{\nearrow} & & \searrow\!\!\!= & \\ A & & & & Q^*. \end{array}$$

It remains to be proved that this really gives an isomorphism.

First a general remark: suppose that Q^* is an acyclic complex of T-acyclic objects of $\mathfrak{A}$. Then TQ^* is also acyclic, since we have $TQ^* = \underline{\underline{R}}TQ^*$ and the latter object is isomorphic to zero in $D(k)$, because Q^* is acyclic and therefore isomorphic to zero in $D(\mathfrak{A})$. Thus

$$\operatorname{Hom}_{K(\mathfrak{A})}(A, Q^*) \cong h^0 \operatorname{Hom}_{\mathfrak{A}}(A, Q^*) = h^0 T(Q^*) = 0.$$

Now let $Q^* \xrightarrow{\simeq} Q'^*$ be a quasi-isomorphism of complexes of T-acyclics and complete it with the mapping-cone to a distinguished triangle,

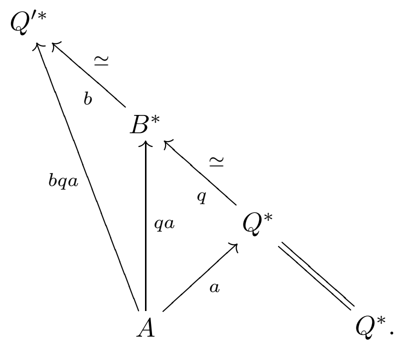
$$\begin{array}{ccc} Q^* & \xrightarrow{\quad\simeq\quad} & Q'^* \\ & \nwarrow \;\; \swarrow & \\ & Q''^*. & \end{array}$$

The mapping-cone is an acyclic complex of T-acyclics, so the long-exact sequence induced by the triangle reads

$$\begin{array}{l} 0 = \operatorname{Hom}_{K(\mathfrak{A})}(A, Q''^*(-1)) \\ \qquad\qquad \downarrow \\ \operatorname{Hom}_{K(\mathfrak{A})}(A, Q^*) \longrightarrow \operatorname{Hom}_{K(\mathfrak{A})}(A, Q'^*) \longrightarrow \operatorname{Hom}_{K(\mathfrak{A})}(A, Q''^*) = 0, \end{array}$$

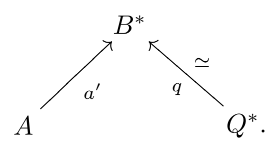
whence $\mathrm{Hom}_{K(\mathfrak{A})}(A, Q^*) \xrightarrow{\cong} \mathrm{Hom}_{K(\mathfrak{A})}(A, Q'^*)$ is an isomorphism.

φ is injective: Assume that $\varphi(a) = 0$. By [4, proof of prop. I.3.1, p. 30], this means that we can choose $q : Q^* \xrightarrow{\simeq} B^*$ such that qa is null-homotopic. But choose $b : B^* \xrightarrow{\simeq} Q'^*$ with Q'^* consisting of T-acyclics. We then have the diagram



bqa is null-homotopic because qa is, but since bq is a quasi-isomorphism, by the general remark above, we must have a null-homotopic.

φ is surjective: Suppose that we are given a morphism α in $\mathrm{Hom}_{D(\mathfrak{A})}(A, Q^*)$. It is represented by a diagram



We can construct a resolution $b : B^* \xrightarrow{\simeq} Q'^*$ where Q'^* consists of T-acyclics. But then bq is a quasi-isomorphism from Q^* to Q'^*, so by the general remark above we can find $a : A \longrightarrow Q^*$ such that ba' and bqa are homotopy-equivalent. It is easy to see that the class of this a is mapped to the given morphism, α, by φ. $\square$

Proposition 2.4. *Suppose that $d < \infty$. Then the set $\{\mathcal{A}(m)[n]\}_{m,n\in\mathbb{Z}}$ is generating for $D(\mathcal{A})$, i.e., if $\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{A}(m)[n], \mathcal{C}^*) = 0$ for all m and n, then $\mathcal{C}^* = 0$. Also, the set consists of compact $D(\mathcal{A})$-objects, i.e., each $\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{A}(m)[n], -)$ commutes with small direct sums. Thus, since the triangulated category $D(\mathcal{A})$ has small direct sums, it is compactly generated, cf.* [6, def. 1.7, p. 211].

Proof. The first claim follows directly from lemma 2.2, since an acyclic complex is isomorphic to the zero-complex in the derived category $D(\mathcal{A})$.

The second claim is an equally direct consequence of lemma 2.3: by [6, observation after def. 1.6, p. 210] it will suffice to prove that each $\mathcal{A}(m)$ is compact, and if we set $T(-) = \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{A}(m), -)$ we know by proposition 2.1 that $\underline{\underline{R}}T$ exists and preserves small direct sums. But then

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{A}(m), -) \cong h^0 \underline{\underline{R}}T(-)$$

also preserves small direct sums. $\square$

3. Non-commutative Serre-duality

This section contains two general results, theorem 3.1 and corollary 3.3, and their corollaries, 3.5 (hypercohomological Serre-duality) and 3.6 (classical Serre-duality).

Throughout this section we assume that the cohomological dimension, d, of Γ is finite.

Theorem 3.1. *The derived functor $\underline{\underline{R}}\Gamma$ has a right-adjoint,*

$$G : D(k) \longrightarrow D(\mathcal{A}).$$

Proof. This is an immediate consequence of the Brown Adjoint Functor Theorem, cf. [6, thm. 4.1, p. 223], since we have already proved propositions 2.1 and 2.4. $\square$

Definition 3.2. *We set*

$$\omega^* = G(k), \quad \omega^\circ = h^{-d}(\omega^*).$$

ω^* will be called the dualizing complex.

Corollary 3.3. *There is an isomorphism*

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{C}^*, \omega^*) \cong (h^0 \underline{\underline{R}}\Gamma\mathcal{C}^*)'$$

natural in $\mathcal{C}^* \in D(\mathcal{A})$; as in the introduction, the prime denotes dualization with respect to k.

Proof. Theorem 3.1 states that there is a natural isomorphism

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{C}^*, G\mathcal{V}^*) \cong \mathrm{Hom}_{D(k)}(\underline{\underline{R}}\Gamma\mathcal{C}^*, \mathcal{V}^*)$$

in $\mathcal{C}^* \in D(\mathcal{A})$ and $\mathcal{V}^* \in D(k)$. In particular, with $\mathcal{V}^* = k$,

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{C}^*, \omega^*) \cong \mathrm{Hom}_{D(k)}(\underline{\underline{R}}\Gamma\mathcal{C}^*, k).$$

But k is an injective resolution of itself, so

$$\mathrm{Hom}_{D(k)}(\underline{\underline{R}}\Gamma\mathcal{C}^*, k) \cong h^0 \, \mathrm{Hom}^*(\underline{\underline{R}}\Gamma\mathcal{C}^*, k) \cong (h^0 \underline{\underline{R}}\Gamma\mathcal{C}^*)'.$$

$\square$

Lemma 3.4. *The only non-vanishing cohomology of ω^* is in degrees $-d, -d+1, \ldots, 0$.*

Proof. The duality-isomorphism yields

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{A}(-m)[-n], \omega^*) &\cong (h^0 \underline{\underline{R}}\Gamma\mathcal{A}(-m)[-n])' \\ &= (R^{-n}\Gamma\mathcal{A}(-m))' = H^{-n}(\mathcal{A}(-m))',\end{aligned}$$

so the lemma's statement follows directly from lemma 2.2. $\square$

Lemma 3.4 implies that

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & \omega^{-d-1} & \longrightarrow & \omega^{-d} & \longrightarrow & \omega^{-d+1} & \longrightarrow & \cdots & \longrightarrow & \omega^0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \| & & & & \| & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \omega^{-d}/B^{-d}(\omega^*) & \longrightarrow & \omega^{-d+1} & \longrightarrow & \cdots & \longrightarrow & \omega^0 & \longrightarrow & \cdots
\end{array}$$

is a quasi-isomorphism, whence we may assume that ω^* has $\omega^{-d-r} = 0$ for every $r > 0$. As a consequence, by [4, lemma I.4.6, p. 42] we may assume that ω^* consists of injective Tails(A)-objects. These assumptions will be made henceforth.

Corollary 3.5 (Hypercohomological Serre-duality)**.** *The dualizing complex, ω^*, consisting of injective* Tails(A)*-objects has the property that*

$$H^n(\mathcal{M})' \cong h^{-n} \, \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{M}, \omega^*) = \mathrm{Ext}^{-n}_{\mathrm{Tails}(A)}(\mathcal{M}, \omega^*)$$

functorially in $\mathcal{M} \in$ Tails(A) *(the* Ext *appearing here is hyper-*Ext *over* Tails(A)*).*

Proof. Simple computations tell us that

$$h^0 \underline{\underline{R}}\Gamma(\mathcal{M}[n]) = h^n \underline{\underline{R}}\Gamma\mathcal{M} = H^n(\mathcal{M})$$

and

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{M}[n], \omega^*) &\cong h^0 \mathrm{Hom}^*(\mathcal{M}[n], \omega^*) \\ &= h^{-n} \mathrm{Hom}^*(\mathcal{M}, \omega^*) = h^{-n} \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{M}, \omega^*);\end{aligned}$$

now use corollary 3.3. □

Corollary 3.6 (Classical Serre-duality)**.** *Let $r \in \mathbb{N}$ and assume that*

$$H^{d-1}(\mathcal{A}(-m)) = H^{d-2}(\mathcal{A}(-m)) = \cdots = H^{d-r}(\mathcal{A}(-m)) = 0$$

for all $m \gg 0$ (the condition is vacuous for $r = 0$). Then for $n = d, d-1, \ldots, d-r$ we have

$$H^n(\mathcal{M})' \cong \mathrm{Ext}^{d-n}_{\mathrm{Tails}(A)}(\mathcal{M}, \omega^\circ)$$

functorially in $\mathcal{M} \in$ Tails(A) (the Ext *appearing here is the classical* Ext *over* Tails(A)*).*

Proof. By corollary 3.3 we have

$$\mathrm{Hom}_{D(\mathcal{A})}(\mathcal{A}(-m)[d-n], \omega^*) \cong h^0 \underline{\underline{R}}\Gamma(\mathcal{A}(-m)[d-n])' = H^{d-n}(\mathcal{A}(-m))',$$

so via lemma 2.2, the assumption on the cohomology of the $\mathcal{A}(-m)$'s implies that

$$h^{-d+1}\omega^* = h^{-d+2}\omega^* = \cdots = h^{-d+r}\omega^* = 0.$$

We see that ω^* is the beginning of ω°'s injective resolution, twisted with $[d]$, whence by corollary 3.5

$$H^n(\mathcal{M})' \cong h^{-n} \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{M}, \omega^*) = \mathrm{Ext}^{d-n}_{\mathrm{Tails}(A)}(\mathcal{M}, \omega^\circ).$$

□

Note that the case $r = 0$ yields an isomorphism

$$H^d(\mathcal{M})' \cong \mathrm{Hom}_{\mathrm{Tails}(A)}(\mathcal{M}, \omega^\circ),$$

natural in $\mathcal{M} \in$ Tails(A), without any conditions on A apart from the standing ones, that A is a left noetherian $\mathbb{N}$-graded k-algebra with $d < \infty$.

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