

**A UNIQUE CONTINUATION THEOREM
 FOR THE SCHRÖDINGER EQUATION
 WITH SINGULAR MAGNETIC FIELD**

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ABSTRACT. We show a unique continuation theorem for the Schrödinger equation $(\frac{1}{i}\nabla - \mathbf{A})^2 u + Vu = 0$ with singular coefficients \mathbf{A} and V .

1. MAIN RESULTS

In this paper we show a unique continuation theorem for the Schrödinger operator $H = (\frac{1}{i}\nabla - \mathbf{A})^2 + V$ with singular magnetic field. In fact we shall establish, under some assumptions on \mathbf{A} and V , the following estimate:

$$\int_{B_{2r}(x_o)} |u|^2 dx \leq C \int_{B_r(x_o)} |u|^2 dx.$$

The latter holds for $r > 0$ and $x_o \in \Omega$ with $B_{2r}(x_o) \subset \Omega$ and for solutions $u \in H_{\text{loc}}^2(\Omega)$ of

$$(1) \quad Hu = \left(\frac{1}{i}\nabla - \mathbf{A}\right)^2 u + Vu = 0 \quad \text{in } \Omega \subset \mathbf{R}^n,$$

where $n \geq 3$, Ω is a domain, $\mathbf{A}(x) = (A_j(x))_{j=1}^n$ is real-valued and $V(x) = V^R(x) + iV^I(x)$ is complex-valued. This estimate implies a strong unique continuation theorem (Corollary 1.1). The proof uses a Rellich's type identity and the variational method (originally due to Garofalo and Lin, see e.g., [GL1], [GL2], [Ku1]), which does not need Carleman type estimates. We emphasize that our method requires neither higher integrability nor pointwise estimates for \mathbf{A} compared with previous results [BKRS], [H], [Wo1], [So], [GL1], [GL2], [Ku1].

Throughout this paper we use the notation

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right), \quad V^+(x) = \max(V(x), 0), \quad V^-(x) = \max(-V(x), 0),$$

$$\mathbf{B} = (b_{jk})_{j,k=1}^n, \quad b_{jk}(x) = \frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j},$$

$$D_j = \frac{1}{i} \frac{\partial}{\partial x_j} - A_j, \quad D_j^* = -\frac{1}{i} \frac{\partial}{\partial x_j} - A_j,$$

$$H_{\text{loc}}^m(\Omega) = \{u \in L_{\text{loc}}^2(\Omega); \quad D^\alpha u \in L_{\text{loc}}^2(\Omega), |\alpha| \leq m\}.$$

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Note that $\overline{D_j u} = D_j^* \bar{u}$ for a complex-valued function u .

To state our main results, first we recall the definition of the Kato class $K_n^{\text{loc}}(\Omega)$.

Definition 1.1. We say $f \in L^1_{\text{loc}}(\Omega)$ belongs to the Kato class $K_n^{\text{loc}}(\Omega)$ if $\lim_{r \rightarrow 0} \eta(r; f\chi_{\Omega_o}) = 0$ for every compact subdomain Ω_o of Ω . Here

$$\eta(r; f\chi_{\Omega_o}) = \sup_{x \in \mathbb{R}^n} \int_{\{|x-y| < r\} \cap \Omega_o} \frac{|f(y)|}{|x-y|^{n-2}} dy.$$

The assumptions on \mathbf{A} and V in this paper are the following.

Assumption (A): Let $x_o \in \Omega$ be fixed.

1. $\mathbf{A} \in L^4_{\text{loc}}(\Omega)$, $\nabla \mathbf{A} \in L^2_{\text{loc}}(\Omega)$, $|\mathbf{A}|^2 \in K_n^{\text{loc}}(\Omega)$, $(|x-x_o||\mathbf{B}|)^2 \in K_n^{\text{loc}}(\Omega)$;
2. $V \in K_n^{\text{loc}}(\Omega)$, $(|x-x_o||V^I|)^2 \in K_n^{\text{loc}}(\Omega)$, $(2V^R + (x-x_o) \cdot \nabla V^R)^- \in K_n^{\text{loc}}(\Omega)$.

Taking an arbitrary compact subdomain $\Omega_o \subset \Omega$ such that $x_o \in \Omega_o$, we will use the notation

$$\begin{aligned} \theta_o(r) &= \eta_o(r; (2V^R + (x-x_o) \cdot \nabla V^R)^-) \\ &\quad + \eta_o(r; ((x-x_o)V^I)^2)^{1/2} + \eta_o(r; ((x-x_o)|\mathbf{B}|)^2)^{1/2}, \end{aligned}$$

where $\eta_o(r; f) = \eta(r; f\chi_{\Omega_o})$.

Theorem 1.1. Suppose Assumption (A) and $\int_0^{r_o} \frac{\theta_o(r)}{r} dr < +\infty$ for some $r_o > 0$. Let $u \in H^2_{\text{loc}}(\Omega)$ be a solution of (1). Then there exist constants $r_*, C > 0$ such that

$$(2) \quad \int_{B_{2r}(x_o)} |u|^2 dx \leq C \int_{B_r(x_o)} |u|^2 dx$$

for every $0 < r < r_*$.

When we do not assume $\int_0^{r_o} \frac{\theta_o(r)}{r} dr < +\infty$, we have

Theorem 1.2. Suppose Assumption (A) and let $u \in H^2_{\text{loc}}(\Omega)$ be a solution of (1). Then, for each $r_1 \in (0, r_*)$, there exist constants $C > 0$ and $L(r_1) > 0$ such that

$$(3) \quad \int_{B_{2r}(x_o)} |u|^2 dx \leq C \exp\left(\frac{L(r_1)}{r^{C\theta_o(r_1)}}\right) \int_{B_r(x_o)} |u|^2 dx$$

for every $r_1/2 > r > 0$.

The constant r_* is determined by the condition that $\eta(r; (V^R)^-) \leq \frac{1}{2C(n)}$ hold for all $r \in (0, r_*)$, where $C(n)$ is a constant depending only on n (Lemma 2.3). The constants C and $L(r_1)$ do not depend on r , but depend on u . It is well known that these yield unique continuation theorems (e.g., [GL1], [GL2], [Ku1]).

Corollary 1.1. Suppose Assumption (A) and the condition $\int_0^{r_o} \frac{\theta_o(r)}{r} dr < +\infty$, $r_o > 0$, for every $x_o \in \Omega$. Then H has SUCP (strong unique continuation property); if $u \in H^2_{\text{loc}}(\Omega)$ is a solution of (1) and satisfies, for some $x_o \in \Omega$ and for every $m > 0$,

$$\int_{B_r(x_o)} |u|^2 dx = O(r^m) \quad (r \rightarrow 0),$$

then $u \equiv 0$ in Ω .

Corollary 1.2. *Suppose Assumption (A) for every $x_o \in \Omega$. Then H has a unique continuation property; if $u \in H_{loc}^2(\Omega)$ is a solution of (1) and satisfies, for some $x_o \in \Omega$ and $A, \alpha > 0$,*

$$\int_{B_r(x_o)} |u|^2 dx = O(\exp(-\frac{A}{r^\alpha})) \quad (r \rightarrow 0),$$

then $u \equiv 0$ in Ω .

In particular, H has WUCP (weak unique continuation property); if u vanishes on a subdomain Ω' of Ω , then $u \equiv 0$ in Ω .

Since $Hu = -\Delta u + 2i\mathbf{A} \cdot \nabla u + i(\operatorname{div}\mathbf{A})u + |\mathbf{A}|^2u + Vu$, putting $\mathbf{b} = 2i\mathbf{A}$, $W = i(\operatorname{div}\mathbf{A}) + |\mathbf{A}|^2 + V$, we can rewrite the equation (1) in the following way:

$$(4) \quad Hu = -\Delta u + \mathbf{b} \cdot \nabla u + Wu.$$

Although we can apply results of [BKRS], [Wo1], [Wo2], [So], [H] [GL2], [Ku1] etc. under suitable conditions on \mathbf{b} and W , our theorems cannot be covered by these previous results. First, previous results require a stronger condition on $\operatorname{div}\mathbf{A}$ even for WUCP. For instance, [Wo2] require $W \in L_{loc}^{n/2}$ for WUCP and hence $\operatorname{div}\mathbf{A} \in L_{loc}^{n/2}$ etc. On the other hand, our method only requires $\operatorname{div}\mathbf{A} \in L_{loc}^2$ for $\operatorname{div}\mathbf{A}$ and, instead of that, $(|x - x_o||\mathbf{B}|)^2 \in K_n^{loc}(\Omega)$ (or $(|x - x_o||\mathbf{B}|)^2 \in F_t^{loc}$) for the magnetic field \mathbf{B} (see Example 1.1). Secondly, to obtain SUCP [BKRS], [H] require $|\mathbf{A}| \in L_{loc}^q, q > \frac{3n-2}{2}$ (for related results see also [Wo1]) and [So], [GL2], [Ku1] require a pointwise estimate

$$|\mathbf{A}(x)| \leq \frac{f(|x - x_o|)}{|x - x_o|}, \quad \int_0^{r_o} \frac{f(t)}{t} dt < +\infty$$

for \mathbf{A} , but our method requires neither higher integrability nor pointwise estimates.

Remark 1.1. These theorems also hold even if we replace the class $K_n^{loc}(\Omega)$ in Assumption (A) by the more general one $Q_t^{loc}(\Omega) = K_n^{loc}(\Omega) + F_t^{loc}(\Omega)$, $1 < t \leq n/2$, where $F_t^{loc}(\Omega)$ is the Fefferman-Phong class. However, in this case we must assume an additional condition

$$\limsup_{r \rightarrow 0} \sup_{x \in \Omega} \|(V^R)^-\|_{Q_t(B_r(x) \cap \Omega)} \leq \epsilon_o$$

for sufficiently small $\epsilon_o > 0$ and take $H_{loc}^2(\Omega) \cap L_{loc}^\infty(\Omega)$ as a solution class. See [Ku1].

Remark 1.2. A sufficient condition to assure $\int_0^{r_o} \frac{\theta_o(r)}{r} dr < +\infty$ is that

$$(2V^R + (x - x_o) \cdot \nabla V^R)^-, ((x - x_o)V^I)^2, ((x - x_o)|\mathbf{B}|)^2 \in K_{n,\delta}^{loc}(\Omega) \subset K_n^{loc}(\Omega)$$

hold for some $\delta > 0$, where we say $f \in K_{n,\delta}^{loc}(\Omega)$ if f satisfies, for every compact subdomain Ω_o of Ω ,

$$\limsup_{r \rightarrow 0} \sup_{x \in \mathbb{R}^n} \int_{\{|x-y| < r\} \cap \Omega_o} \frac{|f(y)|}{|x-y|^{n-2+\delta}} dy = 0.$$

Remark 1.3. By using the approximation argument in [Ku1] we can show unique continuation theorems similar to the theorems above even for H_{loc}^1 solutions.

Finally, we must remark that Kalf [Ka] proved WUCP under the assumptions $\mathbf{A} \in H_{loc}^1(\Omega)$, $(V^R)^2 \in K_n^{loc}(\Omega)$ ($V^I \equiv 0$), but his method cannot be applied to SUCP. For WUCP our results complement his result; compare that one needs

$(V^R)^2 \in K_n^{\text{loc}}(\Omega)$ in his case and $(2V^R + (x - x_o) \cdot \nabla V^R)^- \in K_n^{\text{loc}}(\Omega)$ in our case (see Examples 1.2, 1.3).

Example 1.1. Let $A_i = C \frac{x_i}{|x|}$. Then $\mathbf{B} \equiv 0$ and $\nabla \mathbf{A} \in L_{\text{loc}}^2$, $\mathbf{A} \in L_{\text{loc}}^4$ when $n \geq 5$. Note that $|\mathbf{A}|^2 = C^2/|x|^2 \notin K_n^{\text{loc}}$, but $|\mathbf{A}|^2 = C^2/|x|^2 \in F_t^{\text{loc}}$ for $1 < t < n/2$. Hence we can apply our method to show SUCP for this \mathbf{A} (see Remark 1.1).

Example 1.2. Let $x_o = O, B_1 = B_1(O)$ and

$$V(x) = \frac{1}{|x|^l |\log |x||^m}, \quad m, l > 0.$$

Then we have

$$\begin{aligned} V \in K_n^{\text{loc}}(B_1) &\Leftrightarrow (m > 1, l = 2) \text{ or } (m \in \mathbf{R}^1, l < 2), \\ (5) \quad V^2 \in K_n^{\text{loc}}(B_1) &\Leftrightarrow (m > 1/2, l = 1) \text{ or } (m \in \mathbf{R}^1, l < 1), \\ 2V + x \cdot \nabla V \in K_n^{\text{loc}}(B_1) &\Leftrightarrow (m > 0, l = 2) \text{ or } (m \in \mathbf{R}^1, l < 2). \end{aligned}$$

So in this example the condition $V^2 \in K_n^{\text{loc}}(\Omega)$ is stronger than $(2V + x \cdot \nabla V)^- \in K_n^{\text{loc}}(\Omega)$.

Example 1.3. Let $N \in \mathbf{N}$ and $R = (R_1, \dots, R_N) \in \mathbf{R}^{3N}$ be fixed and

$$V(x) = \sum_{j=1}^N \frac{1}{|x_j - R_j|}, \quad x = (x_1, \dots, x_N) \in \mathbf{R}^{3N}, \quad x_j \in \mathbf{R}^3.$$

Then $V^2 \notin K_n^{\text{loc}}$, but $2V + (x - x_o) \cdot \nabla V \in K_n^{\text{loc}}$ for every $x_o \in \mathbf{R}^{3N}$.

2. PROOF OF THE THEOREMS

We may assume x_o is the origin O and write $B_r = B_r(O)$. For the sake of simplicity, we also use $\Omega, \eta(r; f)$ and $\theta(r)$ instead of $\Omega_o, \eta(r; f\chi_{\Omega_o})$ and $\theta_o(r)$, respectively.

Let $u \in H_{\text{loc}}^2(\Omega)$ be a solution of (1) and put

$$(6) \quad I(r) = \int_{B_r} |Du|^2 + V^R |u|^2 dx = \int_{B_r} |(\nabla - i\mathbf{A})u|^2 + V^R |u|^2 dx,$$

$$(7) \quad H(r) = \int_{\partial B_r} |u|^2 dS, \quad N(r) = \frac{rI(r)}{H(r)}.$$

Note that $H = (\frac{1}{i}\nabla - \mathbf{A})^2 + V = \sum_{j=1}^n D_j D_j + V$. Our argument is based on the following identity.

Lemma 2.1. *Suppose Assumption (A) (for \mathbf{A}). Then $u \in H_{\text{loc}}^2(\Omega) \cap L_{\text{loc}}^\infty(\Omega)$ satisfies*

$$\begin{aligned} & \text{Im} \left(\int_{B_r} (x \cdot \overline{Du}) \sum_{j=1}^n D_j D_j u dx \right) \\ &= \frac{r}{2} \int_{\partial B_r} |Du|^2 dS - \left(\frac{n-2}{2} \right) \int_{B_r} |Du|^2 dx \\ (8) \quad & - r \int_{\partial B_r} |n \cdot Du|^2 dS + \text{Re} \left(\sum_{j,k=1}^n \int_{B_r} b_{jk}(x) x_j D_k u \bar{u} dx \right) \end{aligned}$$

for every $r > 0$ with $B_r \subset \Omega$.

Proof. This is a kind of Rellich’s identity. The detailed computation can be seen in [EK]. So we omit it. \square

We remark that under the assumption $|\mathbf{A}|^2, V \in K_n^{\text{loc}}(\Omega)$ a solution $u \in H_{\text{loc}}^1(\Omega)$ is locally bounded (see [Ku2]). So we can apply Lemma 2.1 for solutions $u \in H_{\text{loc}}^2(\Omega)$ of (1). This identity implies

Lemma 2.2. *Suppose Assumption (A). Then for a.e. $r \in (0, R_o)$*

$$\begin{aligned}
 I'(r) &= \frac{n-2}{r} I(r) + 2 \int_{\partial B_r} |n \cdot Du|^2 dS \\
 &- \frac{n-2}{r} \int_{B_r} V^R |u|^2 dx + \int_{\partial B_r} V^R |u|^2 dS \\
 &- \frac{2}{r} \text{Im} \left(\int_{B_r} (x \cdot \overline{Du}) V u dx \right) \\
 (9) \quad &- \frac{2}{r} \text{Re} \left(\sum_{j,k=1}^n \int_{B_r} b_{jk}(x) x_j (D_k u) \overline{u} dx \right)
 \end{aligned}$$

holds, where $R_o = \max\{r; B_r \subset \Omega\}$.

Proof. Note that

$$I'(r) = \int_{\partial B_r} |Du|^2 + V^R |u|^2 dS \quad \text{a.e. } r.$$

Lemma 2.1 and this identity imply (9). \square

Since $x \cdot \overline{Du} = x \cdot D^* \overline{u} = i(x \cdot \nabla \overline{u}) - (x \cdot \mathbf{A}) \overline{u}$, we have

$$\text{Im} \left(\int_{B_r} (x \cdot \overline{Du}) V u dx \right) = \text{Re} \left(\int_{B_r} (x \cdot \nabla \overline{u}) V^R u dx \right) + \text{Im} \left(\int_{B_r} (x \cdot \overline{Du}) V^I u dx \right).$$

Put

$$\begin{aligned}
 J(V^R; u; r) &= \int_{\partial B_r} V^R |u|^2 dS - \frac{n-2}{r} \int_{B_r} V^R |u|^2 dx \\
 (10) \quad &- \frac{1}{r} \int_{B_r} (x \cdot \nabla(|u|^2)) V^R dx.
 \end{aligned}$$

By integration by parts we have

$$J(V^R; u; r) \geq -\frac{1}{r} \int_{B_r} (2V^R + x \cdot \nabla V^R) |u|^2 dx.$$

Then from this observation and Lemma 2.2 we obtain the following estimate:

$$\begin{aligned}
 I'(r) &\geq \frac{n-2}{r} I(r) + 2 \int_{\partial B_r} |n \cdot Du|^2 dS \\
 (11) \quad &- \frac{1}{r} \int_{B_r} (2V^R + x \cdot \nabla V^R) |u|^2 dx \\
 &- \frac{2}{r} \int_{B_r} (|x| |V^I| + |x| |\mathbf{B}|) |Du| |u| dx
 \end{aligned}$$

for a.e. $r \in (0, R_o)$. To control the last two terms we need the following inequality.

Lemma 2.3. *Let $W \in K_n^{\text{loc}}(\Omega)$ and $u \in H_{\text{loc}}^1(\Omega)$. Then there exists a constant $C(n)$ such that*

$$(12) \quad \int_{B_r} |W||u|^2 dx \leq C(n)\eta(r; W\chi_{\Omega_o}) \left(\int_{B_r} |Du|^2 dx + \frac{1}{r} \int_{\partial B_r} |u|^2 dS \right)$$

for $B_r \subset \Omega_o, \overline{\Omega_o} \subset \Omega$.

Proof. Since $|u| \in H_{\text{loc}}^1(\Omega)$, we know that

$$(13) \quad \int_{B_r} |W||u|^2 dx \leq C(n)\eta(r; W\chi_{\Omega_o}) \left(\int_{B_r} |\nabla|u||^2 dx + \frac{1}{r} \int_{\partial B_r} |u|^2 dS \right)$$

holds ([FGL]). Noting $|\nabla|u|| \leq |Du|$ a.e., we obtain the desired inequality. \square

Lemma 2.4. *There exists a constant $r_* > 0$ such that $H(r) > 0$ for every $r \in (0, r_*)$ unless $u \equiv 0$ in B_{r_*} .*

Proof. Note that the constant $r_* > 0$ is determined by the condition $\eta_-^R(r) \leq \frac{1}{2C(n)}$ for $r \in (0, r_*)$, where $\eta_-^R(r) = \eta(r; (V^R)^-)$ and $C(n)$ is the constant in Lemma 2.3. Since $V^R \in K_n^{\text{loc}}(\Omega)$, the constant $r_* > 0$ satisfying this condition exists.

Suppose $H(r_o) = 0$ for some $r_o \in (0, r_*)$. Noting a simple computation yields

$$(14) \quad I(r) = \text{Re} \left(\int_{\partial B_r} u_\rho \bar{u} dS \right) = \text{Re} \left(\int_{\partial B_r} (n \cdot Du) \bar{u} dS \right),$$

we have $I(r_o) = 0$. Since Lemma 2.3 implies

$$(15) \quad \begin{aligned} I(r) &\geq \int_{B_r} |Du|^2 dx - \int_{B_r} (V^R)^- |u|^2 dx \\ &\geq \int_{B_r} |Du|^2 dx - C(n)\eta_-^R(r) \left(\frac{H(r)}{r} + \int_{B_r} |Du|^2 dx \right), \end{aligned}$$

the choice of r_* yields

$$(16) \quad 0 = I(r_o) \geq \frac{1}{2} \int_{B_{r_o}} |Du|^2 dx.$$

Hence we obtain $|Du(x)| = 0$ a.e. $x \in B_{r_o}$. Since $|\nabla|u|| \leq |Du|$ a.e., $|u|$ is constant in B_{r_o} . $H(r_o) = 0$ implies $|u| \equiv 0$ in B_{r_o} . The conclusion can be obtained by an argument similar to the one in [Ku1, Theorem 1.5]. \square

Hence we may assume $H(r) > 0$ for every $r \in (0, r_*)$.

Lemma 2.5. *There exists an absolute constant $C_0 > 0$ such that*

$$(17) \quad \int_{B_r} |Du|^2 dx \leq C_0 I(r)$$

for every $r \in \Gamma = \{r \in (0, r_*); N(r) > 1\}$.

Proof. Since $\frac{H(r)}{r} < I(r)$ for $r \in \Gamma$, (15) implies

$$\int_{B_r} |Du|^2 dx \leq \frac{1 + C(n)\eta_-^R(r)}{1 - C(n)\eta_-^R(r)} I(r) \leq 3I(r)$$

for all $r \in \Gamma$. This concludes the desired estimate. \square

Using (11) and Lemmas 2.3, 2.5 we obtain, for a.e. $r \in \Gamma$,

$$(18) \quad I'(r) \geq \frac{n-2}{r}I(r) + 2 \int_{\partial B_r} |n \cdot Du|^2 dS - C \frac{\theta(r)}{r}I(r),$$

$$\theta(r)\eta(r; (2V^R + x \cdot \nabla V^R)^-) + \eta(r; (|x|V^I)^2)^{1/2} + \eta(r; (|x||\mathbf{B}|)^2)^{1/2}.$$

On the other hand, we have the following identity:

$$(19) \quad H'(r) = \frac{n-1}{r}H(r) + 2\text{Re}(\int_{\partial B_r} u_\rho \bar{u} dS), \quad u_\rho = \frac{x}{r} \cdot \nabla u.$$

Hence (14) and (19) imply

$$(20) \quad \begin{aligned} H'(r) &= \frac{n-1}{r}H(r) + 2I(r), \\ \frac{d}{dr}(\log \frac{H(r)}{r^{n-1}}) &= 2 \frac{I(r)}{H(r)} = 2 \frac{N(r)}{r}. \end{aligned}$$

Therefore, for a.e. $r \in \Gamma$ we obtain

$$(21) \quad \begin{aligned} \frac{N'(r)}{N(r)} &= \frac{1}{r} + \frac{I'(r)}{I(r)} - \frac{H'(r)}{H(r)} \\ &\geq 2 \left(\frac{\int_{\partial B_r} |n \cdot Du|^2 dS}{I(r)} - \frac{I(r)}{H(r)} \right) - C \frac{\theta(r)}{r} \\ &\geq -C \frac{\theta(r)}{r}. \end{aligned}$$

In the last inequality we used Schwarz's inequality. This differential inequality yields the following growth estimate for $N(r)$.

Theorem 2.1. (i) If $\int_0^{r_0} \frac{\theta(r)}{r} dr < +\infty$ for some $r_0 > 0$, then

$$(22) \quad N(r) \leq \max(1, N(r_*)) \exp(C \int_0^{r_*} \frac{\theta(r)}{r} dr)$$

for every $r \in (0, r_*)$.

(ii)

In general, for each $r_1 \in (0, r_*)$ there exist constants $C, L(r_1) > 0$ such that

$$(23) \quad N(r) \leq \frac{L(r_1)}{r^{C\theta(r_1)}}$$

for every $r \in (0, r_1/2)$.

This theorem implies Theorems 1.1 and 1.2 by a standard argument. For the details see [Ku1], [GL2].

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