

THE FIRST INSTABILITY INTERVAL FOR HILL EQUATIONS WITH SYMMETRIC SINGLE WELL POTENTIALS

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ABSTRACT. For Hill equations with symmetric single well (or symmetric single barrier) potentials, the first instability interval is absent when and only when the potential is constant.

1. INTRODUCTION

The Hill equation in its standard form is

$$(1) \quad -y''(x) + q(x)y(x) = \lambda y(x),$$

where λ is a parameter, and $q(x)$ is a real-valued, continuous and periodic function of period a . Let y_1 and y_2 be the linearly independent solutions of (1) which satisfy the initial conditions

$$y_1(0, \lambda) = y_2'(0, \lambda) = 1, \quad y_1'(0, \lambda) = y_2(0, \lambda) = 0.$$

The discriminant of (1) is defined by

$$D(\lambda) = y_1(a, \lambda) + y_2'(a, \lambda).$$

It is known (see e.g. [4, 9]) that $\lambda_0, \lambda_1, \lambda_2, \dots$, the zeros of $D(\lambda) - 2$, are the eigenvalues of (1) subject to the periodic boundary conditions $y(0) = y(a)$ and $y'(0) = y'(a)$, while $\mu_0, \mu_1, \mu_2, \dots$, the zeros of $D(\lambda) + 2$, are the eigenvalues of (1) subject to the semi-periodic boundary conditions $y(0) = -y(a)$ and $y'(0) = -y'(a)$. These eigenvalues are interlaced in the following way:

$$(2) \quad \lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \lambda_4 < \dots$$

For $\lambda \in (\lambda_{2m}, \mu_{2m}) \cup (\mu_{2m+1}, \lambda_{2m+1})$, all solutions of (1) are bounded in $(-\infty, \infty)$. For $\lambda \in (-\infty, \lambda_0) \cup (\mu_{2m}, \mu_{2m+1}) \cup (\lambda_{2m+1}, \lambda_{2m+2})$, all nontrivial solutions of (1) are unbounded in $(-\infty, \infty)$. The intervals (μ_{2m}, μ_{2m+1}) and $(\lambda_{2m+1}, \lambda_{2m+2})$ are called the $(2m+1)$ th and $(2m+2)$ th instability intervals, while $(-\infty, \lambda_0)$ is called the zeroth instability interval.

In 1946, G. Borg [3] proved an inverse theorem for a Hill's equation:

Theorem 1.1. *The potential $q(x)$ has period $a/2$ if and only if all the odd instability intervals are absent, or, equivalently, all zeros of $D(\lambda) + 2$ are double zeros.*

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From this remarkable theorem he drew:

Theorem 1.2. *The potential $q(x)$ is a constant if and only if all the instability intervals, except the zero-th, are absent.*

For alternative proofs of these results, see the papers of Hochstadt [5, 6, 7] and Ungar [10].

In this note, we are concerned with the first instability interval for symmetric single well potentials and symmetric single barrier potentials. By a symmetric single well potential on $[0, a]$ we mean a continuous function $q(x)$ on $[0, a]$ which is symmetric about $x = a/2$ and nonincreasing on $[0, a/2]$. The negative of a symmetric single well potential is called a symmetric single barrier potential. The purpose of this note is to prove the following

Theorem 1.3. *Let $q(x)$ be a symmetric single well (or symmetric single barrier) potential on $[0, a]$. Then $q(x)$ is a constant if and only if the first instability interval is absent.*

2. PROOF OF THE RESULT

We denote by Λ_n and ν_n ($n = 0, 1, 2, \dots$), respectively, the eigenvalues of (1) subject to the Dirichlet boundary condition $y(0) = y(a) = 0$ and the Neumann boundary condition $y'(0) = y'(a) = 0$. Then it is known [4] that $\nu_0 \leq \lambda_0$ and for $n = 0, 1, 2, \dots$,

$$(3) \quad \mu_{2n} \leq \Lambda_{2n} \leq \mu_{2n+1}, \quad \lambda_{2n+1} \leq \Lambda_{2n+1} \leq \lambda_{2n+2},$$

and

$$(4) \quad \mu_{2n} \leq \nu_{2n+1} \leq \mu_{2n+1}, \quad \lambda_{2n+1} \leq \nu_{2n+2} \leq \lambda_{2n+2}.$$

Let $\psi_0(x)$ be a normalized eigenfunction corresponding to λ_0 , so that $\psi_0(x)$ is periodic with period a and $\int_0^a |\psi_0(x)|^2 dx = 1$. We first note:

Lemma 2.1. *If $q(x)$ is symmetric about $x = a/2$, then $\lambda_0 = \nu_0$ and $\psi_0(x)$ is a Neumann eigenfunction corresponding to ν_0 .*

Proof. Since $q(x)$ is periodic with period a and symmetric about $x = a/2$, $q(x)$ is an even function. It follows that if $y(x)$ is a solution of (1), then so also is $y(-x)$. In particular, $\psi_0(x)$ and $\psi_0(-x)$ are solutions of

$$-y''(x) + q(x)y(x) = \lambda_0 y(x)$$

which satisfy the periodic boundary conditions. Since λ_0 is a simple eigenvalue, there is a constant c such that $\psi_0(-x) = c\psi_0(x)$. But $\psi_0(x)$ has no zeros in $[0, a]$ (see [4]), so $\psi_0(-x) = \psi_0(x)$, hence $\psi_0'(0) = \psi_0'(a) = 0$. This shows that $\psi_0(x)$ is a Neumann eigenfunction corresponding to λ_0 . Since $\lambda_0 < \mu_0 \leq \nu_1$ by (2) and (4), we conclude that $\lambda_0 = \nu_0$. \square

We note that $\psi_0(x)$ can be chosen so as to be positive for all x . Also, if $\Psi_0(x)$ is a normalized Dirichlet eigenfunction corresponding to Λ_0 , then $\Psi_0(x)$ can be chosen so as to be positive for $x \in (0, a)$.

Lemma 2.2. *If $q(x)$ is symmetric about $x = a/2$, then the function $\Psi_0(x)/\psi_0(x)$ is increasing on $[0, a/2]$ and decreasing on $[a/2, a]$.*

Proof. Define $W(x) = \psi_0(x)\Psi_0'(x) - \psi_0'(x)\Psi_0(x)$ so that

$$(5) \quad \left[\frac{\Psi_0(x)}{\psi_0(x)} \right]' = \frac{W(x)}{\psi_0^2(x)}.$$

By the differential equation and the fact that $\lambda_0 < \Lambda_0$, we have

$$\begin{aligned} W'(x) &= \psi_0(x)\Psi_0''(x) - \psi_0''(x)\Psi_0(x) \\ &= (\lambda_0 - \Lambda_0)\psi_0(x)\Psi_0(x) < 0 \end{aligned}$$

for $x \in (0, a)$. Since $q(x)$ is symmetric about $x = a/2$, the same is true for $\psi_0(x)$ and $\Psi_0(x)$, so $\psi_0'(a/2) = \Psi_0'(a/2) = 0$. Hence $W(a/2) = 0$. It follows that $W(x) > 0$ for $x \in (0, a/2)$ while $W(x) < 0$ for $x \in (a/2, a)$. This together with (5) proves the lemma. \square

Remark. If $q(x)$ is symmetric about $x = a/2$, then so also is $\Psi_0(x)/\psi_0(x)$. Since ψ_0 and Ψ_0 are normalized, it follows from Lemma 2.2 that there are points x_{\pm} with

$$0 < x_- < \frac{a}{2} < x_+ < a, \quad x_- + x_+ = a$$

such that

$$\Psi_0(x) < \psi_0(x) \quad \text{on } (0, x_-) \cup (x_+, a), \quad \Psi_0(x) > \psi_0(x) \quad \text{on } (x_-, x_+).$$

Using ideas of Lavine [8], we now prove:

Lemma 2.3. *If $q(x)$ is a symmetric single well potential on $[0, a]$, then $\Lambda_0 - \lambda_0 \leq \pi^2/a^2$. If $q(x)$ is a symmetric single barrier potential on $[0, a]$, then $\Lambda_0 - \lambda_0 \geq \pi^2/a^2$. In either case, the equality holds if and only if $q(x)$ is a constant.*

Proof. Consider the one-parameter family of potentials $q(x, t) = tq(x)$, $t \in \mathbf{R}$. We denote by $\lambda_0(t)$ and $\psi_0(x, t)$ the first eigenvalue and normalized eigenfunction of the corresponding Hill equation with periodic boundary conditions. Similarly, we use $\Lambda_0(t)$ and $\Psi_0(x, t)$ for Dirichlet boundary condition. As we know, $\lambda_0(0) = 0$ and $\Lambda_0(0) = \pi^2/a^2$. By the standard formula, we have

$$\frac{d}{dt}\lambda_0(t) = \int_0^a q(x)\psi_0^2(x, t) dx, \quad \frac{d}{dt}\Lambda_0(t) = \int_0^a q(x)\Psi_0^2(x, t) dx.$$

So, if $\Gamma(t) = \Lambda_0(t) - \lambda_0(t)$, then

$$\begin{aligned} \frac{d}{dt}\Gamma(t) &= \int_0^a q(x)[\Psi_0^2(x, t) - \psi_0^2(x, t)] dx \\ &= \int_{[0, x_-(t)] \cup [x_+(t), a]} q(x)[\Psi_0^2(x, t) - \psi_0^2(x, t)] dx \\ &\quad + \int_{x_-(t)}^{x_+(t)} q(x)[\Psi_0^2(x, t) - \psi_0^2(x, t)] dx. \end{aligned}$$

Now, suppose $q(x)$ is a symmetric single well potential. Then, by the preceding remark, $q(x) \geq q(x_-(t)) = q(x_+(t))$ and $\Psi_0^2(x, t) - \psi_0^2(x, t) < 0$ on $(0, x_-(t)) \cup (x_+(t), a)$, while $q(x) \leq q(x_-(t))$ and $\Psi_0^2(x, t) - \psi_0^2(x, t) > 0$ on $(x_-(t), x_+(t))$. Thus,

$$\frac{d}{dt}\Gamma(t) \leq q(x_-(t)) \int_0^a [\Psi_0^2(x, t) - \psi_0^2(x, t)] dx = 0.$$

This implies that $\Gamma(1) \leq \Gamma(0)$, i.e., $\Lambda_0 - \lambda_0 \leq \pi^2/a^2$, and the equality holds only if $q(x)$ is a constant. The result for symmetric single barrier potentials follows by a similar argument. \square

We are now ready to prove Theorem 1.3.

Proof. The *only if* part is obvious. To prove the *if* part, we consider only the case where $q(x)$ is a symmetric single well potential since the proof for barrier potentials is similar. Suppose that the first instability interval for $q(x)$ is absent. Then we have $\mu_0 = \Lambda_0 = \nu_1 = \mu_1$ by (3) and (4). Thus, by Lemmas 2.1 and 2.3, $\nu_1 - \nu_0 \leq \pi^2/a^2$. On the other hand, it was shown in [8] (see also [1, 2]) that among symmetric single well potentials the gap $\nu_1 - \nu_0$ is minimized at constant q , i.e., $\nu_1 - \nu_0 \geq \pi^2/a^2$. It follows that $q(x)$ is a constant. This completes the proof. \square

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