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CONJUGATE POINTS IN THE BOTT-VIRASORO GROUP AND THE KDV EQUATION

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ABSTRACT. We study the geometry of a right invariant metric on a central extension $\widehat{\mathcal{D}}(S^1)$ of the diffeomorphism group of a circle (the Bott-Virasoro group) introduced by Ovsienko and Khesin. We obtain an expression for the curvature tensor of this metric and apply it to find conjugate points in $\widehat{\mathcal{D}}(S^1)$.

1. INTRODUCTION

Let $\mathcal{D}^s(S^1)$ be the group of orientation preserving Sobolev H^s diffeomorphisms of the circle. It is well known that the group $\mathcal{D}^s(S^1)$ as well as its Lie algebra of vector fields on S^1, $\mathrm{Vect}^s(S^1) = T_e\mathcal{D}^s(S^1)$, have non-trivial one-dimensional central extensions, the Bott-Virasoro group $\widehat{\mathcal{D}}^s(S^1)$ and the Virasoro algebra $\widehat{\mathrm{Vect}}^s(S^1)$ respectively (see for example Kirillov [K] or Segal [S1]). In what follows we assume that s is a nonnegative integer sufficiently large so that the above sets can be equipped with smooth manifold structures and all formal computations in the paper are justified.

The group multiplication in $\widehat{\mathcal{D}}^s(S^1)$ is given by

$$\hat{\eta} \circ \hat{\xi} = (\eta \circ \xi, \alpha + \beta + b(\eta, \xi)), \tag{1.1}$$

where $\hat{\eta} = (\eta, \alpha)$, $\hat{\xi} = (\xi, \beta)$ with $\eta, \xi \in \mathcal{D}^s(S^1)$ and $\alpha, \beta \in R$, and where b is a cocycle on $\mathcal{D}^s(S^1)$, whose explicit formula was found by Bott [Bo]:

$$b(\eta, \xi) = \int_{S^1} \log \partial_x(\eta \circ \xi)\, d \log \partial_x \xi.$$

On the other hand, the commutator in the Virasoro algebra, $\widehat{\mathrm{Vect}}^s(S^1)$, is given by

$$[\widehat{V}, \widehat{W}] = -\left((v\partial_x w - w\partial_x v)\frac{\partial}{\partial x}, c(v, w) \right), \tag{1.2}$$

where $\widehat{V} = (v\frac{\partial}{\partial x}, a)$, $\widehat{W} = (w\frac{\partial}{\partial x}, b)$ with $a, b \in R$, $v\frac{\partial}{\partial x}$, $w\frac{\partial}{\partial x} \in T_e\mathcal{D}^s(S^1)$, and where c is the corresponding Gelfand-Fuchs [GF] cocycle on the Virasoro algebra,

$$c(v, w) = \int_{S^1} \partial_x^2 v \partial_x w \; dx.$$

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From a recent result of Ovsienko and Khesin [OK] (see also [S2]) it follows that the Euler equation on the Virasoro algebra for the L^2 inner product given at the identity element $\hat{e} = (e, 0)$ by

$$\langle \widehat{V}, \widehat{W} \rangle_{L^2} = \int_{S^1} vw \, dx + ab \tag{1.3}$$

is the KdV equation

$$\partial_t v + 3v\partial_x v + a\partial_x^3 v = 0$$

(here $\widehat{V}$ and $\widehat{W}$ are as above).

Thus solutions to the KdV equation correspond, as in the case of the Euler equations of hydrodynamics (cf. Arnol′d [A] or Ebin and Marsden [EM]), to geodesics in $\widehat{\mathcal{D}}^s(S^1)$ of the right invariant metric which at the identity is given by (1.3). In fact, the initial value problem for the KdV equation with initial data in H^s is globally well posed (even for $s = 0$, cf. [B]), and hence the corresponding geodesics exist for all time.

It is therefore of interest to study the geometry of the Bott-Virasoro group. In this paper we compute the curvature tensor of the metric (1.3) at the identity $\widehat{e} \in \widehat{\mathcal{D}}^s(S^1)$ and show that its sectional curvature can assume positive as well as negative signs. We then use these computations to demonstrate that the geodesics corresponding to the trivial solutions of the KdV equation lie in regions of nonnegative sectional curvature and contain conjugate points.

Namely, we show

Theorem. *Any unit speed geodesic in $\widehat{\mathcal{D}}^s(S^1)$ with initial conditions $\widehat{\eta}(0) = (e, 0)$ and $\dot{\widehat{\eta}}(0) = (c\frac{\partial}{\partial x}, a)$, where c and a are constants, contains points conjugate to $\widehat{\eta}(0)$ along $\widehat{\eta}$.*

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2. Sectional curvature

Extend the metric (1.3) to the whole group $\widehat{\mathcal{D}}^s(S^1)$ by setting for any $\widehat{\xi}$ and $\widehat{V}$, $\widehat{W} \in T_{\hat{\xi}}\widehat{\mathcal{D}}^s(S^1)$

$$\langle \widehat{V}, \widehat{W} \rangle_{\hat{\xi}} = \langle \mathrm{d}_{\hat{\xi}} R_{\hat{\xi}^{-1}} \widehat{V}, \mathrm{d}_{\hat{\xi}} R_{\hat{\xi}^{-1}} \widehat{W} \rangle_{L^2}, \tag{2.1}$$

where from (1.1) the derivative of the right translation is given by

$$\mathrm{d}_{\hat{\xi}} R_{\hat{\zeta}} \widehat{V} = \widehat{V} \circ \widehat{\zeta} \tag{2.2}$$

$$= \frac{\partial}{\partial s}\left(\widehat{\xi}(s) \circ \widehat{\zeta}\right)|_{s=0} = \left((v\frac{\partial}{\partial x}) \circ \zeta, a + \int_{S^1} \frac{1}{\partial_x \xi \circ \zeta} \partial_x v \circ \zeta \, d\log(\partial_x \zeta) \right),$$

for any $\widehat{\xi} = (\xi, \alpha)$, $\widehat{\zeta} = (\zeta, \beta)$ and $\widehat{V} = (v\frac{\partial}{\partial x}, a) \in T_{\hat{\xi}}\widehat{\mathcal{D}}^s(S^1)$ with $s \to \widehat{\xi}(s)$ a curve in $\widehat{\mathcal{D}}^s(S^1)$ such that $\widehat{\xi}(0) = \widehat{\xi}$ and $\partial_s \widehat{\xi}(0) = \widehat{V}$.

Let R denote the curvature tensor of the metric (2.1). The following result is also of independent interest.

Proposition 1. *Suppose* $\widehat{V} = (v\frac{\partial}{\partial x}, a)$, $\widehat{W} = (w\frac{\partial}{\partial x}, b)$ *and* $\widehat{U} = (u\frac{\partial}{\partial x}, c)$ *are arbitrary elements of* $T_{\widehat{e}}\widehat{\mathcal{D}}^s(S^1)$, *and suppose* $\widehat{e} = (e, 0)$. *Then*

$$R(\widehat{V},\widehat{W})\widehat{U} \tag{2.3}$$

$$= \frac{1}{4}\Big(\big(c(a\partial_x^6 w - b\partial_x^6 v) + 2u(a\partial_x^4 w - b\partial_x^4 v) + 10\partial_x u(a\partial_x^3 w - b\partial_x^3 v)$$
$$+18\partial_x^2 u(a\partial_x^2 w - b\partial_x^2 v) + 12\partial_x^3 u(a\partial_x w - b\partial_x v) + 2c(v\partial_x^4 w - w\partial_x^4 v) - 2c(\partial_x v\partial_x^3 w - \partial_x w\partial_x^3 v)$$
$$+4u(v\partial_x^2 w - w\partial_x^2 v) + 8\partial_x u(v\partial_x w - w\partial_x v) + \partial_x^3 v\int_{S^1}\partial_x^2 w\partial_x u dx - \partial_x^3 w\int_{S^1}\partial_x^2 v\partial_x u dx$$
$$-2\partial_x^3 u\int_{S^1}\partial_x^2 v\partial_x w dx\big)\frac{\partial}{\partial x},$$
$$\int_{S^1}\big(\partial_x^3 u(a\partial_x^3 w - b\partial_x^3 v) - 2\partial_x u(w\partial_x^3 v - 2\partial_x w\partial_x^2 v + 2\partial_x v\partial_x^2 w - v\partial_x^3 w)\big)dx\Big);$$

$$\langle R(\widehat{V},\widehat{W})\widehat{W},\widehat{V}\rangle_{L^2} \tag{2.4}$$

$$= \int_{S^1}\left(\frac{1}{4}(b\partial_x^3 v - a\partial_x^3 w + 2w\partial_x v - 2v\partial_x w)^2 - 9\partial_x v\partial_x w(b\partial_x^2 v + a\partial_x^2 w)\right)dx$$
$$-\frac{3}{4}\left(\int_{S^1}\partial_x^2 v\partial_x w dx\right)^2.$$

Proof. First, extend the vectors $\widehat{V}$, $\widehat{W}$ and $\widehat{U}$ to right invariant vector fields, also denoted $\widehat{V}$, $\widehat{W}$ and $\widehat{U}$, on the group $\widehat{\mathcal{D}}^s(S^1)$. Let $\mathrm{ad}^*_{\widehat{V}}$ denote the adjoint of $\mathrm{ad}_{\widehat{V}}$ with respect to the metric (2.1). Using (1.2), (1.3) and periodicity of $v(x)$, $w(x)$ and $u(x)$ and integrating by parts, we get

$$\langle \mathrm{ad}^*_{\widehat{V}}\widehat{W},\widehat{U}\rangle_{L^2} = \langle \widehat{W},[\widehat{V},\widehat{U}]\rangle_{L^2} = -\int_{S^1}\big(wv\partial_x u - wu\partial_x v + b\partial_x^2 v\partial_x u\big)dx$$
$$= \langle\big((b\partial_x^3 v + 2w\partial_x v + v\partial_x w)\frac{\partial}{\partial x}, 0\big),\widehat{U}\rangle_{L^2}.$$

Next, observe that since for any right invariant vector fields $\widehat{V}$ and $\widehat{W}$, their inner product $\langle\widehat{V},\widehat{W}\rangle$ is a constant function on $\widehat{\mathcal{D}}^s(S^1)$, the covariant derivative $\nabla_{\widehat{V}}\widehat{W}$ can be obtained explicitly from the formula (see e.g. [CE])

$$2\nabla_{\widehat{V}}\widehat{W} = [\widehat{V},\widehat{W}] - \mathrm{ad}^*_{\widehat{V}}\widehat{W} - \mathrm{ad}^*_{\widehat{W}}\widehat{V} \tag{2.5}$$
$$= -\Big((b\partial_x^3 v + a\partial_x^3 w + 4v\partial_x w + 2w\partial_x v)\frac{\partial}{\partial x}, \int_{S^1}\partial_x^2 v\partial_x w dx\Big).$$

Using (2.5), we now compute

$$4\nabla_{\widehat{V}}\nabla_{\widehat{W}}\widehat{U}$$
$$= \Big(\big(ca\partial_x^6 w + ba\partial_x^6 u + 2au\partial_x^4 w + 10a\partial_x u\partial_x^3 w + 18a\partial_x^2 u\partial_x^2 w + 14a\partial_x^3 u\partial_x w$$
$$+4aw\partial_x^4 u + 4cv\partial_x^4 w + 4bv\partial_x^4 u + 2c\partial_x v\partial_x^3 w + 2b\partial_x v\partial_x^3 u + 16vw\partial_x^2 u + 8vu\partial_x^2 w$$
$$+24v\partial_x w\partial_x u + 8w\partial_x u\partial_x v + 4u\partial_x w\partial_x v + \partial_x^3 v\int_{S^1}\partial_x^2 w\partial_x u dx\big)\frac{\partial}{\partial x},$$
$$\int_{S^1} -\partial_x^3 v\big(c\partial_x^3 w + b\partial_x^3 u + 4w\partial_x u + 2u\partial_x w\big)dx\Big).$$

$\nabla_{\widehat{W}}\nabla_{\widehat{V}}\widehat{U}$ is obtained from the above by interchanging $\widehat{V}$ and $\widehat{W}$, while combining (2.5) and (1.2) gives

$$\begin{aligned}2\nabla_{[\widehat{V},\widehat{W}]}\widehat{U} \\ &= \Big(\big(c(v\partial_x^4 w - w\partial_x^4 v) + 2c(\partial_x v\partial_x^3 w - \partial_x w\partial_x^3 v) \\ &\quad + 2u(v\partial_x^2 w - w\partial_x^2 v) + 4\partial_x u(v\partial_x w - w\partial_x v) \\ &\quad + \partial_x^3 u\int_{S^1}\partial_x^2 v\partial_x w dx\big)\frac{\partial}{\partial x}, \int_{S^1}\partial_x^3 u(v\partial_x w - w\partial_x v)dx\Big).\end{aligned}$$

(2.3) follows now directly from the definition of the curvature tensor, $R(\widehat{V},\widehat{W})\widehat{U} = \nabla_{\widehat{V}}\nabla_{\widehat{W}}\widehat{U} - \nabla_{\widehat{W}}\nabla_{\widehat{V}}\widehat{U} - \nabla_{[\widehat{V},\widehat{W}]}\widehat{U}$.

Finally, expression (2.4) for the sectional curvature follows from (2.3) after a straightforward (if lengthy) integration by parts. □

Corollary 2. *Let $a, c \in R$ and $\widehat{V} = (c\frac{\partial}{\partial x}, a)$. Then*

$$\langle R(\widehat{V},\widehat{W})\widehat{W},\widehat{V}\rangle_{L^2} = \frac{1}{4}\int_{S^1}(a\partial_x^3 w + 2c\partial_x w)^2 dx \geq 0$$

for any $\widehat{W} = (w\frac{\partial}{\partial x}, b) \in T_{\hat{e}}\widehat{\mathcal{D}}^s(S^1)$.

In particular, if $\widehat{V} = (0, a)$ belongs to the centre of the Virasoro algebra $T_{\hat{e}}\widehat{\mathcal{D}}^s(S^1)$, then the sectional curvatures evaluated at $\widehat{V}$ are non-negative. Furthermore, substituting in this case $w(x) = \sin x$ into the formula in Corollary 2 above gives the positive number $\frac{a^2\pi}{4}$.

On the other hand one can also find two planes at the identity for which the sectional curvature is negative.

Corollary 3. *Let $\widehat{V} = (c\sin x\frac{\partial}{\partial x}, a)$ and $\widehat{W} = (d\cos x\frac{\partial}{\partial x}, b)$. Then*

$$\langle R(\widehat{V},\widehat{W})\widehat{W},\widehat{V}\rangle_{L^2} = (\frac{a^2}{c^2} + \frac{b^2}{d^2} + 8 - 3\pi)\frac{c^2d^2}{4}\pi.$$

Clearly, a suitable choice of a, b, c and d gives a negative number.

Remark 1. The curvature computations presented above suggest that it may be possible to study stability of the initial value problem for the KdV equation as the problem of geodesic deviation on the Bott-Virasoro group, as in the case of hydrodynamics (cf. [A] or [M]).

3. Proof of the theorem

In order to find conjugate points it will be convienient to study the Jacobi equation on the Virasoro algebra. Let $\widehat{\eta}(t)$ be the geodesic with initial conditions given by the theorem. Further, let $\widehat{W}(t)$ be an arbitrary vector field along $\widehat{\eta}$ and set

$$(w(t,x)\frac{\partial}{\partial x}, b(t)) := \mathrm{d}_{\hat{\eta}_t}R_{\hat{\eta}_t^{-1}}\widehat{W}(t).$$

It is straightforward to verify that $\widehat{\eta}(t)(x) = (x+ct, at)$ and that from (2.2) we have $\mathrm{d}_{\hat{\eta}_t}R_{\hat{\eta}_t^{-1}}\dot{\widehat{\eta}}(t) = \dot{\widehat{\eta}}(0)$. The right invariance of the metric $\langle\cdot,\cdot\rangle$, and hence its curvature tensor R, on $\widehat{\mathcal{D}}^s(S^1)$, together with formula (2.2) and the curvature expression (2.3) of Proposition 1, yield

$$\mathrm{d}_{\hat{\eta}_t} R_{\hat{\eta}_t^{-1}} \left(R(\widehat{W}(t), \dot{\widehat{\eta}}(t)) \dot{\widehat{\eta}}(t) \right) = R\left((w(t,x)\frac{\partial}{\partial x}, b(t)), (c\frac{\partial}{\partial x}, a) \right) (c\frac{\partial}{\partial x}, a)$$

$$= -\frac{1}{4} \left((a^2 \partial_x^6 w + 4ac\partial_x^4 w + 4c^2 \partial_x^2 w) \frac{\partial}{\partial x}, 0 \right).$$

Similarly, from (2.5)

$$\mathrm{d}_{\hat{\eta}_t} R_{\hat{\eta}_t^{-1}} \left(\nabla_{\dot{\hat{\eta}}_t} \nabla_{\dot{\hat{\eta}}_t} \widehat{W}(t) \right) = \nabla_{(c\frac{\partial}{\partial x}, a)} \nabla_{(c\frac{\partial}{\partial x}, a)} (w(t,x)\frac{\partial}{\partial x}, b(t))$$

$$= \nabla_{(c\frac{\partial}{\partial x}, a)} \left((\partial_t w + 2c\partial_x w + \frac{1}{2} a \partial_x^3 w) \frac{\partial}{\partial x}, \partial_t b \right)$$

$$= \left((\partial_t^2 w + 4c\partial_t \partial_x w + 4c^2 \partial_x^2 w + a\partial_t \partial_x^3 w + 2ac\partial_x^4 w + \frac{1}{4} a^2 \partial_x^6 w) \frac{\partial}{\partial x}, \partial_t^2 b \right).$$

Combining the two expressions above we find that the Jacobi equation along $\widehat{\eta}$ on the Virasoro algebra has the form

$$\begin{aligned} \partial_t^2 w + 4c\partial_t \partial_x w + 3c^2 \partial_x^2 w + a\partial_t \partial_x^3 w + ac\partial_x^4 w &= 0, \\ \partial_t^2 b &= 0. \end{aligned} \tag{3.1}$$

Now, for any integer $n \geq 1$ such that $an^2 \neq 2c$, a direct inspection shows that the pair

$$w(t,x) = \sin \frac{(an^2 - 2c)n}{2} t \sin\big(nx + \frac{(an^2 - 4c)n}{2} t\big),$$

$$b(t) = 0,$$

is a non-trivial solution of (3.1) and therefore

$$\widehat{W}(t)(x) = \sin \frac{(an^2 - 2c)n}{2} t \left(\sin\big(nx + \frac{(an^2 - 2c)n}{2} t\big) \frac{\partial}{\partial x}, 0 \right)$$

is a non-zero Jacobi field along $\widehat{\eta}(t)$. It is clearly perpendicular to $\widehat{\eta}$ and vanishes at the points

$$t = \frac{2\pi}{(an^2 - 2c)n} k,$$

for $k = 0, \pm 1, \pm 2, \ldots$. For example, one may pick $n = 2$ if $a = 2c = \pm 2/\sqrt{2(\pi + 2)}$ and $n = 1$ otherwise. The theorem follows.

Remark 2. In particular, setting $c = 0$, $a = 1$ in the theorem shows that the points $(x, 2\pi k)$, $k = \pm 1, \pm 2, \ldots$, are conjugate to $(x, 0)$ along the geodesic issued in the direction of the central element of the Virasoro algebra.

Remark 3. Equation (3.1) can also be derived directly on the Virasoro algebra from the second variation formula using the reduction method of [MR] or [BKMR].

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