

**FIXED POINT THEORY FOR COMPACT  
UPPER SEMI-CONTINUOUS OR LOWER SEMI-CONTINUOUS  
SET VALUED MAPS**

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ABSTRACT. Fixed point theory is presented for compact u.s.c. and l.s.c. set valued maps.

1. INTRODUCTION

This paper presents new fixed point theory for compact upper semi-continuous (u.s.c.) and lower semi-continuous (l.s.c.) set valued mappings. Our results generalize some well known fixed point results for convex and nonconvex maps. Our paper is divided into two main sections. In section 2 we use a recent Leray–Schauder type alternative due to Ben–El–Mechaiekh and Idzik [3] for u.s.c., compact approachable maps with nonempty, compact values to establish a new fixed point result for such maps. Our theory generalizes some fixed point results in Ben–El–Mechaiekh and Deguire [2] and Granas [9]. In section 3 we establish via Michael’s selection theorem [11] a nonlinear alternative of Leray–Schauder type for l.s.c., compact multifunctions with closed, convex values. Also a new fixed point result is given. This generalizes some results in Himmelberg, Porter and Van Vleck [10] and Tarafdar and Vybórný [17]. A brief discussion of nonconvex maps is also given.

We now gather together some definitions and known facts. Let  $X$  and  $Y$  be topological spaces. A multifunction  $f : X \rightarrow Y$  is a point to set function such that for each  $x \in X$ ,  $f(x)$  is a nonempty subset of  $Y$ . The function  $f$  is u.s.c. if the set  $f^{-1}(B) = \{x \in X : f(x) \cap B \neq \emptyset\}$  is closed for any closed set  $B$  in  $Y$  (equivalently,  $f : X \rightarrow Y$  is u.s.c. if for any net  $\{x_\alpha\}$  in  $X$  and any closed set  $B$  in  $Y$  with  $x_\alpha \rightarrow x_0 \in X$  and  $f(x_\alpha) \cap B \neq \emptyset$  for all  $\alpha$  we have  $f(x_0) \cap B \neq \emptyset$ ). The function  $f : X \rightarrow Y$  is l.s.c. if the set  $f^{-1}(A)$  is open for any open set  $A$  in  $Y$ .  $f : X \rightarrow Y$  is compact if  $f(X)$  is relatively compact in  $Y$ . Finally the function  $f$  is said to be point-compact if for each  $x \in X$ ,  $f(x)$  is a compact subset of  $Y$ . Now we state a result [16, proposition 1] which will be used frequently in this paper.

**Theorem 1.1.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be a u.s.c., point-compact multifunction. Suppose  $\{x_\alpha\}$  is a net in  $X$  such that  $x_\alpha \rightarrow x_0$ . If  $y_\alpha \in f(x_\alpha)$  for each  $\alpha$ , then there is a  $y_0 \in f(x_0)$  and a subnet  $\{y_\beta\}$  of the net  $\{y_\alpha\}$  such that  $y_\beta \rightarrow y_0$ .*

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Theorem 1.1 immediately yields the following well known result.

**Theorem 1.2.** *Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be a u.s.c., point-compact multifunction. Suppose  $\{x_\alpha\}$  is a net in  $X$  and  $y_\alpha \in f(x_\alpha)$  for each  $\alpha$ . If  $x_\alpha \rightarrow x_0$  and  $y_\alpha \rightarrow y_0$ , then  $y_0 \in f(x_0)$ .*

## 2. THE UPPER SEMI-CONTINUOUS CASE

We begin by describing the class of approachable set valued maps. Let  $X$  and  $Y$  be subsets of Hausdorff topological vector spaces  $E_1$  and  $E_2$  respectively, and let  $F : X \rightarrow Y$  be a multifunction. Given two open neighborhoods  $U$  and  $V$  of the origins in  $E_1$  and  $E_2$  respectively, a  $(U, V)$ -continuous selection of  $F$  is a continuous function  $s : X \rightarrow Y$  satisfying

$$s(x) \in (F[(x + U) \cap X] + V) \cap Y \quad \text{for every } x \in X.$$

$F$  is said to be approachable if it has a  $(U, V)$ -continuous selection for any open neighborhoods  $U$  and  $V$  of the origins in  $E_1$  and  $E_2$  respectively. Let

$$\mathcal{A}_0(X, Y) = \{F : X \rightarrow Y : F \text{ is approachable}\}$$

and

$$\mathcal{A}(X, Y) = \{F \in \mathcal{A}_0(X, Y) : F \text{ is u.s.c. and compact valued}\}.$$

*Remark.* For examples of approachable maps we refer the reader to [2, 3].

We now gather together three theorems from the literature which will be used in this section.

**Theorem 2.1** ([2], pg.486, proposition 3.3). *Let  $X, Y$  and  $Z$  be subsets of the Hausdorff topological vector spaces  $E_1, E_2$  and  $E_3$  respectively. Suppose  $F : X \rightarrow Y$  is a multivalued map with  $F \in \mathcal{A}(X, Y)$  and  $r : Z \rightarrow X$  is a continuous map (single valued). If  $Z$  is compact then  $Fr \in \mathcal{A}(Z, Y)$ .*

**Theorem 2.2** ([2], pg.496, corollary 7.3). *Let  $X$  be a convex subset of a locally convex Hausdorff linear topological space  $E$  with  $F \in \mathcal{A}(X, X)$  a compact map. Then  $F$  has a fixed point (i.e. there exists  $x \in X$  with  $x \in F(x)$ ).*

**Theorem 2.3** ([3], pg.108, Theorem). *Let  $X$  be a closed subset of a locally convex Hausdorff linear topological space  $E$  with  $0 \in \text{int } X$ . In addition assume  $F : X \rightarrow E$  is a u.s.c., approachable, compact map with nonempty, closed values. Then either*

- (A1)  $F$  has a fixed point in  $X$ ; or
- (A2) there exists  $(\lambda, x) \in (0, 1) \times \partial X$  such that  $x \in \lambda F(x)$ .

We now state and prove the main result of this section.

**Theorem 2.4.** *Let  $C$  be a complete convex subset of a metrizable (metric  $d$ ) locally convex linear topological space  $E$  with  $Q$  a closed, convex, proper subset of  $C$  and  $0 \in Q$ . Assume  $F : Q \rightarrow C$  is a u.s.c., approachable, compact map with nonempty, compact values. In addition suppose*

$$(2.1) \quad \begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^\infty \text{ is a sequence in } \partial Q \times [0, 1] \text{ converging to} \\ (x, \lambda) \text{ with } x \in \lambda F(x) \text{ and } 0 \leq \lambda < 1, \\ \text{then there exists } j_0 \in \{1, 2, \dots\} \text{ with} \\ \{\lambda_j F(x_j)\} \subseteq Q \text{ for each } j \geq j_0. \end{cases}$$

Finally assume either

$$(2.2a) \quad \begin{cases} C \text{ is compact and } U_i = \{x \in E : d(x, Q) < \frac{1}{i}\} \subseteq C \\ \text{for } i \text{ sufficiently large,} \end{cases}$$

or

$$(2.2b) \quad Q \subseteq \overline{\text{co}}(\overline{F(Q)}) \text{ and } U_i \subseteq \overline{\text{co}}(\overline{F(Q)}) \text{ for } i \text{ sufficiently large,}$$

or

$$(2.2c) \quad \begin{cases} Fr \in \mathcal{A}_0(C, C) \text{ and } U_i \subseteq C \text{ for } i \text{ sufficiently large; here} \\ r : E \rightarrow Q \text{ is a continuous retraction such that} \\ r(z) \in \partial Q \text{ for } z \in E/Q \end{cases}$$

is satisfied. Then  $F$  has a fixed point in  $Q$ .

*Remarks.* (i) The existence of a continuous retraction  $r : E \rightarrow Q$  follows immediately from Dugundji's extension theorem [7, 18].

(ii) If  $0 \in \text{int } Q$  we may take

$$r(x) = \frac{x}{\max\{1, \mu(x)\}}, \quad x \in E.$$

Here  $\mu$  is the Minkowski functional [15] on  $Q$ , i.e.  $\mu(x) = \inf\{\alpha > 0 : x \in \alpha Q\}$ . If  $\text{int } Q = \emptyset$  then  $\partial Q = Q$ . As a result we may (and do) choose the retraction  $r$  above so that  $r(z) \in \partial Q$  if  $z \in E/Q$ .

(iii) Notice (2.2a) and (2.2b) are only of interest if the space is normable.

(iv) If  $F$  is convex valued then of course  $Fr$  is convex valued, so  $Fr$  is approachable [3], i.e. (2.2c) holds.

(v) Theorem 2.4 was proved by Furi and Pera [8], by a different method, when  $F$  is single valued, compact and continuous. It was extended by O'Regan [13, 14] for the case when  $F$  is single valued, condensing ( $P$ -concentrative) and continuous.

*Proof.* Let  $Z = C$  if (2.2a) or (2.2c) holds, and let  $Z = \overline{\text{co}}(\overline{F(Q)})$  if (2.2b) is satisfied.

*Remark.* Note  $\overline{\text{co}}(\overline{F(Q)})$  is compact. To see this let  $P$  be a defining system of seminorms. Fix  $p \in P$ . Since  $\overline{F(Q)}$  is compact then  $\overline{\text{co}}(\overline{F(Q)})$  is precompact (totally bounded) [6] in the seminormed space  $(E, p)$  for each  $p \in P$ . Thus  $\overline{\text{co}}(\overline{F(Q)})$  is precompact in  $E$ . Now since  $\overline{\text{co}}(\overline{F(Q)})$  is a closed precompact subset of  $C$  and  $C$  is complete, then  $\overline{\text{co}}(\overline{F(Q)})$  is compact [5].

Now let

$$B = \{x \in Z : x \in Fr(x)\}.$$

We first show  $B \neq \emptyset$ . Notice  $Fr : Z \rightarrow Z$  is a u.s.c. (since  $F$  is u.s.c. and  $r$  is continuous), approachable (theorem 2.1 if (2.2a) or (2.2b) occurs and by assumption if (2.2c) occurs), compact map with nonempty, compact values. Theorem 2.2 implies that  $Fr$  has a fixed point, so  $B \neq \emptyset$ . We next claim that  $B$  is closed. Let  $(x_\alpha)$  be a net in  $B$  with  $x_\alpha \rightarrow x_0 \in Z$ . Theorem 1.2 implies  $x_0 \in Fr(x_0)$ , so  $x_0 \in B$ . Thus  $B$  is closed, and in fact  $B$  is compact since  $B \subseteq Fr(B) \subseteq F(Q)$ .

It remains to show  $B \cap Q \neq \emptyset$ . To do this we argue by contradiction. Suppose  $B \cap Q = \emptyset$ . Then since  $B$  is compact and  $Q$  is closed there exists  $\delta > 0$  with  $\text{dist}(B, Q) > \delta$ . Choose  $m \in \{1, 2, \dots\}$  such that  $1 < \delta m$  and  $U_i \subseteq Z$  for  $i \in \{m, m+1, \dots\}$ ; here  $U_i = \{x \in E : d(x, Q) < \frac{1}{i}\}$ . Fix  $i \in \{m, m+1, \dots\}$ . Since  $\text{dist}(B, Q) > \delta$ , then  $B \cap \overline{U_i} = \emptyset$ . Now  $0 \in U_i$  and  $Fr : \overline{U_i} \rightarrow C$  is a u.s.c., approachable, compact map with nonempty, compact values. Theorem 2.3 implies (since  $B \cap \overline{U_i} = \emptyset$ ) that there exists  $(y_i, \lambda_i) \in \partial U_i \times (0, 1)$  with  $y_i \in \lambda_i Fr(y_i)$ . Notice in particular since  $y_i \in \partial U_i$  that

$$(2.3) \quad \{\lambda_i Fr(y_i)\} \not\subseteq Q \quad \text{for each } i \in \{m, m+1, \dots\}.$$

Now consider

$$D = \{x \in E : x \in \lambda Fr(x) \text{ for some } \lambda \in [0, 1]\}.$$

We first show  $D$  is closed. To see this let  $(x_\alpha)$  be a net in  $D$  (i.e. for each  $\alpha$ ,  $x_\alpha \in \lambda_\alpha Fr(x_\alpha)$  for some  $\lambda_\alpha \in [0, 1]$ ) with  $x_\alpha \rightarrow x_0 \in E$ . Without loss of generality assume  $\lambda_\alpha \rightarrow \lambda_0 \in [0, 1]$ . We claim that

$$N : E \times [0, 1] \rightarrow C \quad \text{given by } N(x, \lambda) = \lambda Fr(x)$$

is u.s.c. with nonempty, compact values. If the claim is true then theorem 1.2 implies  $x_0 \in N(x_0, \lambda_0)$ , i.e.  $x_0 \in \lambda_0 Fr(x_0)$ , so  $D$  is closed. To see that  $N$  is u.s.c. let  $\Omega$  be a closed subset of  $C$ ,  $(y_\alpha, t_\alpha)$  a net in  $E \times [0, 1]$ ,  $(y_\alpha, t_\alpha) \rightarrow (y_0, t_0)$  and  $t_\alpha Fr(y_\alpha) \cap \Omega \neq \emptyset$ . We must show  $t_0 Fr(y_0) \cap \Omega \neq \emptyset$ . Suppose  $w_\alpha \in Fr(y_\alpha)$  with  $t_\alpha w_\alpha \in \Omega$ . Since  $Fr$  is u.s.c., there exist (theorem 1.1)  $w_0 \in Fr(y_0)$  and a subnet  $(w_\beta)$  of  $(w_\alpha)$  with  $w_\beta \rightarrow w_0$ . Since  $\Omega$  is closed we have  $t_0 w_0 \in \Omega$ . Consequently  $t_0 Fr(y_0) \cap \Omega \neq \emptyset$ , so  $N : E \times [0, 1] \rightarrow C$  is u.s.c. Thus  $D$  is closed and also compact since  $D \subseteq \text{co}(Fr(Q) \cup \{0\})$ . This together with  $d(y_j, Q) = \frac{1}{j}$ ,  $|\lambda_j| \leq 1$  (for  $j \in \{m, m+1, \dots\}$ ), implies that we may assume without loss of generality that  $\lambda_j \rightarrow \lambda^*$  and  $y_j \rightarrow y^* \in \partial Q$ . Also, since  $y_j \in \lambda_j Fr(y_j)$ , we have from theorem 1.2 that  $y^* \in \lambda^* Fr(y^*)$  (note from above that  $N : \overline{U_m} \times [0, 1] \rightarrow C$  given by  $N(u, \lambda) = \lambda Fr(u)$  is u.s.c. with nonempty, compact values). If  $\lambda^* = 1$ , then  $y^* \in Fr(y^*)$ , which contradicts  $B \cap Q = \emptyset$ . Hence we may assume  $0 \leq \lambda^* < 1$ . But in this case, (2.1) with  $x_j = r(y_j) \in \partial Q$ ,  $x = y^* = r(y^*)$  implies that there exists  $j_0 \in \{1, 2, \dots\}$  with  $\{\lambda_j Fr(y_j)\} \subseteq Q$  for each  $j \geq j_0$ . This contradicts (2.3). Thus  $B \cap Q \neq \emptyset$ , so there exists  $x \in Q$  with  $x \in Fr(x)$ , i.e.  $x \in F(x)$ .  $\square$

Let  $X$  and  $Y$  be subsets of a locally convex Hausdorff linear topological space  $E$ . More generally, let

$$\mathcal{B}_0(X, Y) = \{F : X \rightarrow Y : F \text{ satisfies property } B\}$$

and let  $\mathcal{B}(X, Y)$  be the set of maps  $F : X \rightarrow Y$  which are u.s.c., compact with nonempty, compact values and satisfy a property  $B$  (which we will not specify). Suppose the following properties are satisfied:

$$(2.4) \quad \text{if } X \text{ is convex and } F \in \mathcal{B}(X, Y) \text{ then } F \text{ has a fixed point in } X$$

and

$$(2.5) \quad \begin{cases} \text{if } X \text{ is closed and convex, } 0 \in \text{int } X \text{ and } F \in \mathcal{B}(X, E) \\ \text{then either} \\ (A1). F \text{ has a fixed point in } X; \text{ or} \\ (A2). \text{ there exists } (\lambda, x) \in (0, 1) \times \partial X \text{ such that } x \in \lambda F(x). \end{cases}$$

With the above properties, essentially the same reasoning as in theorem 2.4 establishes the following result.

**Theorem 2.5.** *Let  $C$  be a complete convex subset of a metrizable (metric  $d$ ) locally convex linear topological space  $E$  with  $Q$  a closed, convex, proper subset of  $C$ ,  $0 \in Q$  and  $U_i = \{x \in E : d(x, Q) < \frac{1}{i}\} \subseteq C$  for  $i$  sufficiently large. Assume  $F \in \mathcal{B}(Q, C)$ , (2.1) holds and also that*

$$(2.6) \quad \begin{cases} Fr \in \mathcal{B}_0(C, C); \text{ here } r : E \rightarrow Q \text{ is a continuous retraction} \\ \text{such that } r(z) \in \partial Q \text{ for } z \in E/Q \end{cases}$$

*is satisfied. Then  $F$  has a fixed point in  $Q$ .*

*Remark.* For examples of classes of maps  $\mathcal{B}_0(X, Y)$  we refer the reader to [9]. For example, property  $B$  could be that the values of the maps are  $R_\delta$  sets.

### 3. THE LOWER SEMI-CONTINUOUS CASE

We begin by establishing a nonlinear alternative of Leray–Schauder type for l.s.c. set valued maps.

**Theorem 3.1.** *Let  $Q$  be either (i) a closed convex subset of a Fréchet space (complete metrizable locally convex linear topological space)  $E$ , or (ii) a complete, convex, compact, metrizable subset of a locally convex Hausdorff linear topological space  $E$ . Assume  $U$  is a relatively open subset of  $Q$ ,  $0 \in U$  and  $U$  is convex. Suppose  $F : \bar{U} \rightarrow Q$  is a l.s.c., compact map with closed, convex values. Then either*

- (A1)  $F$  has a fixed point in  $\bar{U}$ ; or
- (A2) there exist  $u \in \partial U$  and  $\lambda \in (0, 1)$  with  $u \in \lambda F(u)$ .

*Remark.*  $\bar{U}$  and  $\partial U$  denote the closure and boundary of  $U$  in  $Q$  respectively.

*Proof.* Define the retraction  $r : Q \rightarrow \bar{U}$  by

$$r(x) = \frac{x}{\max\{1, \mu(x)\}}, \quad x \in Q,$$

where  $\mu$  is the Minkowski functional [15] on  $\bar{U}$ , i.e.  $\mu(x) = \inf\{\alpha > 0 : x \in \alpha \bar{U}\}$ . Consider the mapping  $Fr : Q \rightarrow Q$ . Notice  $Fr$  is an l.s.c., compact map with closed, convex values. Michael’s selection theorem [11, theorem 1.2] implies that there exists a continuous function  $f : Q \rightarrow Q$  such that  $f(x) \in Fr(x)$  for all  $x \in Q$ . Next Tychonoff’s theorem [7] implies that  $f$  has a fixed point, which is also of course a fixed point of  $Fr$ , i.e. there exists  $y \in Q$  with  $y \in Fr(y)$ . Thus if  $z = r(y) \in \bar{U}$  we have  $z \in rF(z)$ , i.e.  $z = r(w)$  for some  $w \in F(z)$ , i.e.  $z \in \bar{U}$  is a fixed point of  $rF$ . Now either  $w \in \bar{U}$  or not. If  $w \in \bar{U}$  then  $r(w) = w$ , so (A1) occurs. If  $w \notin \bar{U}$  then  $z = r(w) = \frac{w}{\mu(w)}$ , so  $z \in \lambda F(z)$  where  $\lambda = \frac{1}{\mu(w)} < 1$ , i.e. (A2) occurs. □

*Remark.* In theorem 3.1 conditions were put on  $Q, E$  and  $F$  so that Michael’s selection theorem could be applied. Of course results could be established if we used other selection theorems for l.s.c. set valued maps (convex and nonconvex); see [1, 4, 12] for example. This remark also applies to the next theorem.

We now prove a new fixed point result for l.s.c., compact set valued maps.

**Theorem 3.2.** *Let  $C$  be either (i) a closed convex subset of a Fréchet space (metric  $d$ )  $E$  or (ii) a complete, convex, compact subset of a metrizable (metric  $d$ ) locally convex linear topological space  $E$ . Assume  $Q$  is a closed, convex, proper subset of  $C$  with  $0 \in Q$  and  $U_i = \{x \in E : d(x, Q) < \frac{1}{i}\} \subseteq C$  for  $i$  sufficiently large. Suppose  $F : Q \rightarrow C$  is an l.s.c., compact map with closed, convex values, and that (2.1) holds. Then  $F$  has a fixed point.*

*Proof.* Let  $r : E \rightarrow Q$  be a continuous retraction with  $r(z) \in \partial Q$  for  $z \in E/Q$ . By Michael's selection theorem [11] (since  $Fr : C \rightarrow C$  is an l.s.c., compact map with closed, convex values) there exists a continuous function  $f : C \rightarrow C$  such that  $f(x) \in Fr(x)$  for all  $x \in C$ . Consider

$$B = \{x \in C : x = f(x)\}.$$

Now  $B \neq \emptyset$  by Tychonoff's theorem. Also  $B$  is clearly closed and compact, since  $B \subseteq f(B) \subseteq F(Q)$ . It remains to show  $B \cap Q \neq \emptyset$ . Suppose  $B \cap Q = \emptyset$ . Then there exists  $\delta > 0$  with  $\text{dist}(B, Q) > \delta$ . Choose  $m \in \{1, 2, \dots\}$  such that  $1 < \delta m$  and  $U_i \subseteq C$  for  $i \in \{m, m+1, \dots\}$ . Fix  $i \in \{m, m+1, \dots\}$ . Since  $\text{dist}(B, Q) > \delta$ , then  $B \cap \overline{U_i} = \emptyset$ . Now  $f : \overline{U_i} \rightarrow C$  is a continuous, compact map, so by the nonlinear alternative for single valued maps [7] there exists (since  $B \cap \overline{U_i} = \emptyset$ )  $(y_i, \lambda_i) \in \partial U_i \times (0, 1)$  with  $y_i = \lambda_i f(y_i)$ . Now since  $y_i \in \partial U_i$  we have  $\lambda_i f(y_i) \notin Q$  for  $i \in \{m, m+1, \dots\}$ , and so

$$(3.1) \quad \{\lambda_i Fr(y_i)\} \not\subseteq Q \quad \text{for each } i \in \{m, m+1, \dots\}.$$

Let

$$D = \{x \in C : x = \lambda f(x) \text{ for some } \lambda \in [0, 1]\}.$$

Since  $D$  is compact and  $d(y_j, Q) = \frac{1}{j}$ ,  $|\lambda_j| \leq 1$  (for  $j \in \{m, m+1, \dots\}$ ), we may assume without loss of generality that  $\lambda_j \rightarrow \lambda^*$  and  $y_j \rightarrow y^* \in \partial Q$ . Also, since  $y_j = \lambda_j f(y_j)$  we have  $y^* = \lambda^* f(y^*)$ . Now  $\lambda^* \neq 1$  since  $B \cap Q = \emptyset$ . Thus  $0 \leq \lambda^* < 1$  and  $y^* \in \lambda^* Fr(y^*)$ . But in this case (2.1) with  $x_j = r(y_j)$ ,  $x = y^* = r(y^*)$  implies that there exists  $j_0 \in \{1, 2, \dots\}$  with  $\{\lambda_j Fr(y_j)\} \subseteq Q$  for each  $j \geq j_0$ . This contradicts (3.1), so  $B \cap Q \neq \emptyset$ .  $\square$

*Remark.* One could use the ideas in this section together with ideas in [14] to obtain new results of Leray–Schauder and Furi–Pera type for l.s.c., condensing ( $P$ -concentrative) maps.

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