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ON ISAACS' THREE CHARACTER DEGREES THEOREM

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ABSTRACT. Isaacs has proved that a finite group G is solvable whenever there are at most three characters of pairwise distinct degrees in $\mathrm{Irr}(G)$ (Isaacs' three character degrees theorem). In this note, using Isaacs' result and the classification of the finite simple groups, we prove the solvability of G whenever $\mathrm{Irr}(G)$ contains at most three monolithic characters of pairwise distinct degrees. §2 contains some additional results about monolithic characters.

§1.

Let $\mathrm{Irr}(G)$ be the set of all irreducible characters of a finite group G, and let $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. Isaacs' theorem claims that G is solvable whenever $|\mathrm{cd}(G)| \le 3$.

A group G is said to be a monolith if it contains only one minimal normal subgroup. It is convenient to consider the group $\{1\}$ as a monolith. A character χ of G is said to be monolithic if $\chi \in \mathrm{Irr}(G)$ and $G/\ker(\chi)$ is a monolith. Obviously, a monolith possesses a faithful irreducible character (of course, this character is monolithic). Let

$$\mathrm{Irr}_m(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi \text{ is monolithic }\},\ \mathrm{cd}_m(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}_m(G)\}.$$

Let p denote a prime number. Denote by $G(p')$ the intersection of kernels of all the nonlinear irreducible characters χ of G such that $p \nmid \chi(1)$. It is easy to see that $p \mid \phi(1)$ for all nonlinear $\phi \in \mathrm{Irr}(PG(p'))$, where $P \in \mathrm{Syl}_p(G)$. Therefore $G(p')$ is p-nilpotent and solvable (see [Isa], Corollary 12.2, and [Ber2], Proposition 9 and the Remark, following it). Let $\Phi(G)$, $\mathrm{F}(G)$ be the Frattini subgroup and the Fitting subgroup of G, respectively. It follows from the properties of the Frattini subgroup that G is p-nilpotent if and only if $G/\Phi(G)$ is p-nilpotent.

Note that all irreducible characters of p-groups are monolithic. $|\mathrm{cd}_m(G)| = 1$ if and only if G is abelian. Note that $\bigcap_{\chi\in\mathrm{Irr}_m(G)} \ker\chi = \{1\}$ (Lemma 2(a)), so that G is a subgroup of a direct product of monoliths. Therefore the set $\mathrm{Irr}_m(G)$ is sufficiently large to have a strong influence on the structure of G. On the other hand, the following examples show that in many cases the set $\mathrm{cd}_m(G)$ is very small compared with $\mathrm{cd}(G)$.

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Examples. 1. Let $G = \mathrm{SL}(2,5) \times \mathrm{L}_2(7)$. Then $|\operatorname{Irr}(G)| = 54$, $|\operatorname{Irr}_m(G)| = 14$.

2. Let $p_1, \dots, p_n$ be pairwise distinct odd primes, G_i a dihedral group of order $2p_i$ $(i = 1, \dots, n)$, and $G = G_1 \times \cdots \times G_n$. Then $|\operatorname{cd}(G)| = n+1$, $|\operatorname{cd}_m(G)| = 2$.

3. Let $p_1, \dots, p_n$ be pairwise distinct odd primes, $4 \mid p_i$ for $i > 1$. Let G_1 be a dihedral group of order $2p_1$, and G_i be a Frobenius group of order $4p_i$ $(i = 2, \dots, n)$. Put $G = G_1 \times \cdots \times G_n$. Then $|\operatorname{cd}(G)| = 2n$, $|\operatorname{cd}_m(G)| = 3$.

These examples show that in many cases $\operatorname{Irr}_m(G)$ is a rather small subset of $\operatorname{Irr}(G)$.

As usual we consider $\operatorname{Irr}(G/N)$ as a subset of $\operatorname{Irr}(G)$.

This paper was inspired by Chapter 12 of Isaacs' book [Isa] and some old results of Isaacs and Passman.

We collected some known results in the following

Lemma 1. *Let N be a normal subgroup of G.*

(a) (Gallagher; [Isa], *Corollary 6.17) Let $\chi \in \operatorname{Irr}(G)$. If $\chi_N \in \operatorname{Irr}(N)$, then $\chi\vartheta \in \operatorname{Irr}(G)$ for all $\vartheta \in \operatorname{Irr}(G/N)$.*

(b) ([Ber1], *Remark 1) If N is nonsolvable, then*

$$|\{\chi(1) \mid \chi \in \operatorname{Irr}(G) - \operatorname{Irr}(G/N)\}| > 1.$$

(c) ([BZ1], *Proposition 30.18) Let p be a prime. If $p \mid \chi(1)$ for every nonlinear $\chi \in \operatorname{Irr}_m(G)$, then G is p-nilpotent and solvable.*

(d) ([BZ1], *Proposition 30.18) $p \nmid \chi(1)$ for every $\chi \in \operatorname{Irr}_m(G)$ if and only if a Sylow p-subgroup is normal in G and abelian.*

(e) (Broline-Garrison, see [Isa], *Theorem 12.19) Let $\chi \in \operatorname{Irr}(G)$ and $K = \ker(\chi)$. Either of the following conditions guarantees the existence of $\psi \in \operatorname{Irr}(G)$ with $\psi(1) > \chi(1)$ and $\ker(\psi) < K$:*

(1e) $K \not\leq \mathrm{F}(G)$.

(2e) $K = \mathrm{F}(G)$, $G/K > \{1\}$ is solvable.

Proof. We prove (c) only. Let G be a counterexample of minimal order. Then G is a monolith. Let R be a minimal normal subgroup of G. Since $Y = \operatorname{Irr}(G) - \operatorname{Irr}(G/R) \subseteq \operatorname{Irr}_m(G)$, it follows that $p \mid \chi(1)$ for all $\chi \in Y$. Hence, $R \leq G(p')$ so that R is solvable. Since by induction G/R is solvable and p-nilpotent and G is a counterexample, R is a p-subgroup. Since G is not p-nilpotent, $R \not\leq \Phi(G)$. Take $P \in \operatorname{Syl}_p(G)$. It follows from the modular law that $R \not\leq \Phi(P)$. In particular, $R \not\leq P'$. Take a linear character λ of P such that $R \not\leq \ker(\lambda)$. Since $p \nmid \lambda^G(1)$, there exists $\chi \in \operatorname{Irr}(\lambda^G)$ such that $p \nmid \chi(1)$. By reciprocity $R \not\leq \ker(\chi)$, so that χ is monolithic. Since $R \leq G'$, it follows that $G' \not\leq \ker(\chi)$, and so $\chi \in \operatorname{Irr}_m(G)$ is nonlinear — a contradiction. $\square$

Lemma 1(c) generalizes Thompson's Theorem (see [Isa], Corollary 12.2). Lemma 1(d) is a generalization of Michler's result [Mic].

Our principal aim is to prove the following.

Theorem. *If $|\operatorname{cd}_m(G)| \leq 3$, then G is solvable.*

Proof. Let G be a counterexample of minimal order. Since $|\operatorname{cd}_m(G/N)| \leq |\operatorname{cd}_m(G)|$, it follows from the induction hypothesis that G is a monolith. Let R be a minimal normal subgroup of G. Then G/R is solvable and $R = R_1 \times \cdots \times R_s$, where $R_1, \dots, R_s$ are isomorphic nonabelian simple groups. Obviously, $R = R' \leq G'$.

If $R = G'$, then every nonlinear irreducible character of G is monolithic. Therefore $|\operatorname{cd}(G)| = |\operatorname{cd}_m(G)| \leq 3$, and G is solvable by Isaacs' three character degrees theorem. Thus,

(i) $R < G'$.

In particular, G/R is nonabelian. Let H/R be a normal subgroup of G/R such that G/H is nonabelian, but every proper epimorphic image of G/H is abelian. By [Isa], Lemma 12.3, G/H is a p-group, or $G/H = (B/H,\ L/H)$ is a Frobenius group with cyclic complement B/H and kernel $L/H = (G/H)'$; obviously, G/H is a monolith. In what follows we fix so-defined subgroups H, B and L.

By [Isa], Lemma 12.3, $\operatorname{cd}(G/H) = \{1,\ t\}$, $t > 1$.

Let ϕ be a nonlinear irreducible character of G/H. Then ϕ is monolithic, so that $\phi(1) = t \in \operatorname{cd}_m(G)$. Obviously, $1 \in \operatorname{cd}_m(G)$. By Lemma 1(b), there exist $\chi,\ \tau \in \operatorname{Irr}(G) - \operatorname{Irr}(G/R)$ such that $\chi(1) \neq \tau(1)$. Obviously, $\chi,\ \tau \in \operatorname{Irr}_m(G)$ and χ, τ are nonlinear. Thus, $\operatorname{cd}_m(G) = \{1,\ \chi(1),\ \tau(1)\}$. We assume that $\chi(1) = t$. Put $\tau(1) = h$. Thus,

(ii) $\operatorname{cd}_m(G) = \{1,\ h,\ t\}$, $1 \neq h \neq t \neq 1$.

We fix so chosen $\phi,\ \chi,\ \tau$.

By Lemma 1(c),

(iii) $(h,\ t) = 1$.

Suppose that G/H is a p-group. Then $t = p^\alpha$, $p \nmid h$, and $\tau_H \in \operatorname{Irr}(H)$ by Clifford theory. Therefore $\tau\phi \in \operatorname{Irr}(G)$ by Lemma 1(a) (recall that ϕ is a nonlinear irreducible character of G/H). Since $R \not\leq \ker(\phi\tau)$, it follows that $\phi\tau \in \operatorname{Irr}_m(G)$, and so $p^\alpha h \in \operatorname{cd}_m(G) = \{1,\ p^\alpha,\ h\}$ — a contradiction. Therefore,

(iv) $G/H = (B/H,\ L/H)$ is a Frobenius group, $L/H = (G/H)'$ is the unique minimal normal subgroup of G/H, B/H is cyclic of order t, and L/H is a p-group, where p is a prime number.

Let a nonprincipal $\psi \in \operatorname{Irr}(R)$, $\vartheta \in \operatorname{Irr}(\psi^G)$. Since $R \not\leq \ker(\vartheta)$, it follows that ϑ is monolithic and $\vartheta(1) \in \{t,\ h\}$. By Clifford's theorem $\psi(1) \mid t$ or $\psi(1) \mid h$. By (iii), $\operatorname{cd}(R) = M \cup N$ is a nontrivial partition such that inclusions $a \in M,\ b \in N$ imply $(a,\ b) = 1$. It follows from the description of irreducible characters of direct products that

(v) $s = 1$, i.e., R is a simple group.

Since χ_R is reducible by Lemma 1(a), we have, by Clifford's theorem,

(vi) There is a nonprincipal character λ in $\operatorname{Irr}(R)$ such that $\lambda(1) \mid t$ and $\lambda(1) < t$. In particular, $t > 3$.

If λ is as in (vi), then $\lambda(1) > 2$ (Klein's theorem; see [Isa], Theorem 14.23), hence we obtain the following stronger inequality:

(vii) $t \geq 6$.

Since G is a monolith, $\mathrm{C}_G(R) = \{1\}$, where $\mathrm{C}_G(R)$ is the centralizer of R in G. Therefore, G/R is a nonabelian subgroup of $A = \operatorname{Out}(R) = \operatorname{Aut}(R)/R$. In particular, A is nonabelian.

In what follows, we use some information about outer automorphism groups of simple groups (see [Gor], §4.15A, and [LPS], Table 2.1).

Let X be a simple group of Lie type defined over a Galois field $\mathrm{GF}(p^e)$. Then $A = \operatorname{Out}(X) = \operatorname{Aut}(X)/X$ is a solvable group with normal subgroups D and DF, where D, the group of diagonal automorphisms, is abelian of order d, and F, the group of field automorphisms, is cyclic of order e. Also, $A/DF \cong \{1\},\ \mathrm{C}(2),\ \mathrm{S}_3$, where $\mathrm{C}(2)$ is cyclic of order 2, and S_3 is the symmetric group of degree 3. Next,

A/D is abelian, unless $A/DF \cong \mathrm{S}_3$. If $A/DF \cong \mathrm{S}_3$, then A/D contains an abelian subgroup of index 2.

If X is sporadic or alternating, then A is abelian. By (i), R is neither sporadic nor alternating.

Putting $q = p^e$ we have to consider as R the following groups (see [LPS], Table 2.1):

$$\mathrm{L}_n(q),\ n \geq 3;\ \mathrm{U}_n(q),\ n \geq 3;\ \mathrm{P}\Omega^{\pm}_{2m}(q),\ m \geq 4;$$

$$\mathrm{E}_6(q),\ 3 \mid q-1;\ {}^2\mathrm{E}_6(q),\ 3 \mid q+1.$$

It follows from (iv) that $t \mid |L/H|-1$. By (vii), $t \geq 6$. Therefore by [LPS], Table 2.1, we have $R \cong \mathrm{L}_n(q),\ n \geq 3$, or $R \cong \mathrm{U}_n(q),\ n \geq 3$. Then $|A| = 2ed,\ |D| = d$ (where $A = \mathrm{Out}(R)$). In the first case $d = (n,\ q-1)$, in the second one $d = (n,\ q+1)$. In particular, $d \leq n$. If DF is abelian (we use for G the same notation as before for X), then $t = 2$ by Ito's theorem on degrees of irreducible characters (see [Isa], Theorem 6.15), contradicting (vii). Thus, DF is nonabelian. In particular (recall that F is cyclic), $D \not\leq \mathrm{Z}(DF)$, $d > 2$. Now, $G/H = T$ is an epimorphic image of a subgroup of A, and T is a Frobenius group with kernel T' of prime order $r \mid d$ by (iv) (recall that in the case under consideration D is cyclic and $A' \leq D$). Then $\mathrm{cd}_m(T) = \{1,\ t\},\ t \mid r-1$. Since $r \leq d \leq n$, it follows that $t \leq n-1$. Therefore, by (vii), $2 < \lambda(1) \leq \frac{n-1}{2}$ and $n \geq 7$ (recall that a nonprincipal $\lambda \in \mathrm{Irr}(\chi_R)$). Obviously, R contains a subgroup $S \cong \mathrm{A}_n$ (recall that a permutation matrix is unitary). Since R is simple, $S \not\leq \ker(\lambda)$, hence $\mathrm{Irr}(\lambda_S)$ contains a nonlinear character μ. Since $\mu(1) \leq \lambda(1) \leq \frac{n-1}{2}$, it follows that $\mathrm{Irr}(\mathrm{A}_n)$ contains a nonlinear irreducible character of degree at most $\frac{n-1}{2}$, but this is impossible (we use the representation theory of the symmetric groups). The proof is completed. □

Conjecture 1. Let N be a normal subgroup of G. If

$$\mathrm{c}(G,\ N) = |\{\chi(1) \mid \chi \in \mathrm{Irr}(G) - \mathrm{Irr}(G/N)\}| < 3,$$

then N is solvable.

Of course, Conjecture 1 implies the Theorem. If N is nonsolvable, then Lemma 1(b) implies $\mathrm{c}(G,\ N) \geq 2$.

Considering a minimal counterexample G to Conjecture 1, we may assume that G is a monolith such that its minimal normal subgroup $R = N$. As in the proof of the Theorem, we can prove that R is simple. Further on, $\mathrm{C}_G(R) = \{1\}$ so G/R is solvable by the classification of the finite simple groups. We consider this Conjecture as a very difficult one.

Conjecture 2. If $|\,\mathrm{cd}_m(G)| \leq 3$, then the derived length of G does not exceed 3 (Proposition 4(d) gives a weaker result in our case).

We note that the derived length of G does not exceed 3 if $|\,\mathrm{cd}(G)| \leq 3$ (Isaacs; see [Isa], Theorem 12.15), but there is yet an unproven conjecture that the inequality $\mathrm{dl}(G) \leq |\,\mathrm{cd}(G)|$ is true for all solvable groups G.

Question 1. Let N be a proper normal subgroup of G. Suppose that

$$|\{\chi(1) \mid \chi \in \mathrm{Irr}_m(G) - \mathrm{Irr}(G/N)\}| = 1.$$

Describe the structure of N and G in detail.

We note that Question 1 is very difficult even if N is a minimal normal subgroup of G (it follows from Lemma 1(b) that N is solvable). Recently the pairs $N \triangleleft G$ were classified such that $|\operatorname{Irr}(G) - \operatorname{Irr}(G/N)| = 1$ (Burtzev and Kazarin; first results about the structure of such G were proved by Gagola [Gag]; see also [BCZ], Lemma 2).

Question 2. Prove the Theorem without using the classification of the finite simple groups.

§2.

In this section we prove some additional results about monolithic characters.

Let $\operatorname{Irr}_1(G)$ be the set of all nonlinear irreducible characters of G. Put

$$\operatorname{Irr}_{1,m}(G) = \operatorname{Irr}_m(G) \cap \operatorname{Irr}_1(G),$$

$$\mathfrak{D}_m(G) = \bigcap_{\chi \in \operatorname{Irr}_m(G)} \ker(\chi),\ \mathfrak{D}_{1,m}(G) = \bigcap_{\chi \in \operatorname{Irr}_{1,m}(G)} \ker(\chi).$$

Since G' is the intersection of the kernels of the linear monolithic characters of G, $\mathfrak{D}_m(G) = \mathfrak{D}_{1,m}(G) \cap G'$.

Lemma 2. *(a)* $\mathfrak{D}_m(G) = \{1\}$.
(b) $\mathfrak{D}_{1,m}(G) \leq \mathrm{Z}(G)$.

Proof. We may assume that $G > \{1\}$. Let R be a minimal normal subgroup of G. Take in G a maximal normal subgroup L such that $R \cap L = \{1\}$. Then G/L is a monolith with minimal normal subgroup RL/L. Let χ be a faithful irreducible character of G/L. Since R is arbitrary and $R \not\leq \ker(\chi)$, it follows that $\mathfrak{D}_m(G) = \{1\}$, and this proves (a). Furthermore, $\mathfrak{D}_{1,m}(G) \cap G' = \mathfrak{D}_m(G) = \{1\}$ by (a). Take $d \in \mathfrak{D}_{1,m}(G)$ and $x \in G$. Then $[x,\ d] \in \mathfrak{D}_{1,m}(G) \cap G' = \{1\}$, and this proves (b). $\square$

The group $G = \mathrm{SL}(2,5) \times \mathrm{C}(3)$, where $\mathrm{C}(3)$ is a cyclic group of order 3, shows that the inclusion $\mathfrak{D}_{1,m}(G) < \mathrm{Z}(G)$ is possible.

Proposition 3. *If G is nonabelian and $|\operatorname{Irr}_{1,m}(G)| = 1$, then one of the following assertions holds:*
(a) $G = \mathrm{ES}(m,\ 2) \times A$, where $\mathrm{ES}(m,\ 2)$ is an extraspecial group of order 2^{1+2m} and A is abelian of odd order.
(b) $G/\mathrm{Z}(G) \cong \mathrm{AGL}(1,\ p^\alpha)$, $G' \cap \mathrm{Z}(G) = \{1\}$, $G' \cong \mathrm{E}(p^\alpha)$, the elementary group of order p^α, $G = G_1 \times P$, where $P \in \mathrm{Syl}_p(\mathrm{Z}(G))$.

Proof. Put $\operatorname{Irr}_{1,m}(G) = \{\chi\}$. Then $\ker(\chi) \leq \mathrm{Z}(G)$ (Lemma 2(b)) and $G/\ker(\chi)$ is a monolith. Let $H/\ker(\chi)$ be a normal subgroup of $G/\ker(\chi)$ such that G/H is nonabelian, but every proper epimorphic image of G/H is abelian. Then G/H is a monolith, so there exists $\tau \in \operatorname{Irr}_{1,m}(G/H)$. By assumption, $\chi = \tau$, hence $H = \ker(\chi)$.

Assume that G/H is nilpotent. As a monolith, G/H is primary. Since $\operatorname{Irr}_{1,m}(G/H) = \operatorname{Irr}_1(G/H)$, we have $G/H \cong \mathrm{ES}(m, 2)$ by [Sei]. Since $H \leq \mathrm{Z}(G)$ (Lemma 2(b)), G is nilpotent. In this case $G = P \times A$, where $P \in \mathrm{Syl}_2(G)$, A is abelian. Since $|\operatorname{Irr}_{1,m}(P)| = |\operatorname{Irr}_{1,m}(G/A)| \leq |\operatorname{Irr}_{1,m}(G)| = 1$, we have $P \cong \mathrm{ES}(m,\ 2)$, and G is a group from (a).

Assume that G/H is nonnilpotent. Then by [Isa], Theorem 12.3, and [Sei], $G/H \cong \mathrm{AGL}(1,\ p^\alpha)$. By Lemma 2(b), $H = \mathrm{Z}(G) = Q \times P$, where $P \in \mathrm{Syl}_p(\mathrm{Z}(G))$ (in fact, $\mathrm{Z}(G/H) = \{1\}$ implies $\mathrm{Z}(G) \le H$). Suppose that $Q = \{1\}$. Then $P = \mathfrak{D}_{1,m}(G)$ and $P \cap G' = \{1\}$ (see the proof of Lemma 2). Then $G' \cong \mathrm{E}(p^\alpha)$ and $G'P = G' \times P$ is abelian. By [Hup], Satz 1.17.4(a), $G = G_1 \times P$. Analogously we consider the case $Q > \{1\}$.

Let $G = G_1 \times P,\ G_1/\mathrm{Z}(G_1) \cong \mathrm{AGL}(1,\ p^\alpha),\ P \in \mathrm{Syl}_p(\mathrm{Z}(G))$. It is easy to check that $|\operatorname{Irr}_{1,m}(G)| = 1$. □

Question 3. Classify the groups G such that $|\operatorname{Irr}_{1,m}(G)| \le 3$.

Question 4. Classify the groups G such that any two distinct characters in $\operatorname{Irr}_{1,m}(G)$ have distinct degrees.

If any two distinct characters in $\operatorname{Irr}_1(G)$ have distinct degrees, then

$$G \in \{\mathrm{ES}(m,\ 2),\ \mathrm{AGL}(1,\ p^\alpha),\ (\mathrm{Q}(8),\ \mathrm{E}(9))\},$$

where $\mathrm{Q}(8)$ is the ordinary quaternion group, $\mathrm{E}(9)$ is elementary of order 9, and (A, B) is a Frobenius group with kernel B and complement A [BCH].

Let $\mathrm{dl}(G)$ and $\mathrm{nl}(G)$ denote the derived length and the nilpotent length of a solvable group G, respectively.

Proposition 4. *(a) If* $|\operatorname{Irr}_{1,m}(G)| \le 4$*, then* G *is solvable, unless* $G = G' \times A$*, where* $G' \cong \mathrm{L}_2(5)$.

(b) If $|\operatorname{cd}_m(G)| = 2$*, then* $\mathrm{dl}(G) = 2$.

(c) If all characters in $\operatorname{Irr}_{1,m}(G)$ *have distinct degrees, then* G *is solvable.*

(d) If G *is solvable, then* $|\operatorname{cd}_m(G)| \ge \mathrm{nl}(G)$.

Proof. Assume that in all cases G is a counterexample of minimal order. Then, in cases (b), (c) and (d), G is a monolith. Let R be a minimal normal subgroup of G. Set $|R| = r^\alpha$ if R is solvable, where r is a prime.

(a) If $\operatorname{Irr}_1(G) = \operatorname{Irr}_{1,m}(G)$, then $G \cong \mathrm{L}_2(5)$ by [Ber3] or [Ber4]. Thus, $\operatorname{Irr}_{1,m}(G) \subset \operatorname{Irr}_1(G)$.

(a.i) Suppose that G is a monolith.

It follows from $\operatorname{Irr}_{1,m}(G) \subset \operatorname{Irr}_1(G)$ that $R < G'$, and so G/R is nonabelian. In particular, $|\operatorname{Irr}_{1,m}(G/R)| \ge 1$.

Assume that R is nonsolvable. Let $\phi_1, \dots, \phi_k \in \operatorname{Irr}_1(R)$ be such that $\phi_1(1) < \cdots < \phi_k(1)$. If $\chi^i \in \operatorname{Irr}(\phi_i^G)$ for all i, then $\chi^1, \dots, \chi^k$ are distinct nonlinear monolithic characters of G by reciprocity. By Isaacs' three character degree theorem (or by the Theorem), $k \ge 3$. It follows from $|\operatorname{Irr}_{1,m}(G)| \le 4$ that $|\operatorname{Irr}_{1,m}(G/R)| = 1$, and so $k = 3$. Therefore, R is simple, and, by Proposition 3, G/R is solvable. Let H/R be a maximal normal subgroup of G/R. Then $|G : H| = p$, a prime number. If, say, $(\chi^1)_H \in \operatorname{Irr}(H)$, then $((\chi^1)_H)^G = \chi_1^1 + \cdots + \chi_p^1$, where $\chi_j^1 \in \operatorname{Irr}_{1,m}(G)$ are distinct of the same degree for all j. Since $\chi^2,\ \chi^3 \notin \{\chi_1^1, \dots, \chi_p^1\}$, it follows that $|\operatorname{Irr}_{1,m}(G)| \ge 3 + 1 + p - 1 > 4$ — a contradiction. Analogously, $(\chi^2)_H,\ (\chi^3)_H$ are reducible. Then $(\chi^i)_H = \lambda_1^i + \cdots + \lambda_p^i$, where $\lambda_j^i \in \operatorname{Irr}(H)$ are distinct of the same degree for all $i,\ j$ (by Clifford theory). Therefore $p \mid \chi^i(1)$ for $i = 1,\ 2,\ 3$. This means that $R \le G(p')$. Since $G(p')$ is solvable, R is solvable as well — a contradiction.

Thus, R is solvable. Since G/R is nonsolvable, we see that $|\operatorname{Irr}_{1,m}(G/R)| \ge 4$ by induction. G as a monolith has a faithful irreducible character; therefore, $|\operatorname{Irr}_{1,m}(G)| \ge |\operatorname{Irr}_{1,m}(G/R)| + 1 \ge 4 + 1 = 5$ — a contradiction.

(a.ii) Suppose that G is not a monolith. Let R_1 be a minimal normal subgroup of G, $R_1 \neq R$.

Suppose that one of the subgroups R, R_1, say R, is nonsolvable. Since $RR_1/R_1 \cong R$ is a minimal normal subgroup of G/R_1, we have, by induction, $R \cong \mathrm{L}_2(5)$. Since $\mathrm{Aut}(R) \cong \mathrm{S}_5$ satisfies $|\mathrm{Irr}_{1,m}(\mathrm{S}_5)| = 5$, it follows that $G = R \times \mathrm{C}_G(R)$. Since $|\mathrm{Irr}_{1,m}(G/\mathrm{C}_G(R))| = 4 = |\mathrm{Irr}_{1,m}(G)|$, we obtain $\mathrm{C}_G(R) = \mathrm{Z}(G)$ by Lemma 2(b), and the result follows.

Thus, we may assume that all minimal normal subgroups of G are solvable. Then by induction, $G/\mathrm{S}(G) \cong \mathrm{L}_2(5)$, and, as in the previous paragraph, $\mathrm{S}(G) = \mathrm{Z}(G)$ (here we use Lemma 2(b)). Obviously, $G'' = G'$. Since, by the above, G' is not a minimal normal subgroup of G, it is a representation group of $\mathrm{L}_2(5)$, i.e., by a known result of I. Schur, $G' \cong \mathrm{SL}(2,5)$. We have $G = G'\mathrm{Z}(G)$. Instead of G we may consider its epimorphic image $\bar{G}$ such that $\mathrm{Z}(G)$ is a nonidentity cyclic 2-group. Without loss of generality we may assume that $\mathrm{Z}(G)$ is a cyclic 2-subgroup. It is easy to check that $|\mathrm{Irr}_{1,m}(G)| \geq |\mathrm{Irr}_{1,m}(\mathrm{SL}(2,5))| = 8$ — a contradiction. The proof of (a) is complete.

(b) The group G is solvable by the Theorem. If G is nilpotent, it is an r-group (recall that R is an r-group), $\mathrm{Irr}_m(G) = \mathrm{Irr}(G)$, so that $\mathrm{dl}(G) = 2$ by Taketa's Theorem (see [Isa], Theorem 5.12). Thus, G is not nilpotent. Let $\mathrm{cd}_m(G) = \{1,\ t\}$. Since G is a monolith, it is not r-nilpotent so, by Lemma 1(c), $r \nmid t$. By Lemma 1(d), $T \in \mathrm{Syl}_r(G)$ is normal in G and abelian. Since G is a monolith, $T = \mathrm{F}(G)$. Since $R \leq G'$, G'/R is abelian by induction, and G' is nonabelian by the induction hypothesis, $G' \not\leq T$ (in particular, G' is nonnilpotent); it follows from the properties of the Frattini subgroup that $R \not\leq \Phi(G)$. Then there exists a maximal subgroup M of G such that $G = MR$, $M \cap R = \{1\}$. If $R < T$, then $\mathrm{N}_G(M \cap T) = G$ and $M \cap T > \{1\}$, and so G is not a monolith. Thus, $R = T = \mathrm{F}(G)$. Let H/R be a normal subgroup of G/R such that G/H is nonabelian, but every proper epimorphic image of G/H is abelian. Then G/H is a monolith so that $\mathrm{cd}_m(G/H) = \mathrm{cd}_m(G) = \{1,\ t\}$. There exists $\chi_1 \in \mathrm{Irr}(G)$ such that $\ker(\chi_1) = H$ (in particular $\chi_1 \in \mathrm{Irr}_{1,m}(G)$). By Lemma 1(e) there exists $\chi_2 \in \mathrm{Irr}(G)$ such that $\chi_2(1) > \chi_1(1) = t$ and $\ker(\chi_2) < H$. By assumption χ_2 is not monolithic so that $R < \ker(\chi_2)$. Since $R = \mathrm{F}(G)$, there exists (by Lemma 1(e)) $\chi_3 \in \mathrm{Irr}(G)$ such that $\chi_3(1) > \chi_2(1) > t$ and $\ker(\chi_3) < \ker(\chi_2)$. By asumption, χ_3 is not monolithic, so that $R < \ker(\chi_3)$. Continuing in this way we obtain an infinite sequence of nonmonolithic irreducible characters $\{\chi_i\}$ $(i = 1, 2, \dots)$ of G/R such that $\chi_1(1) < \chi_2(2) < \dots$ — a contradiction.

(c) By induction, G/R is solvable. If G/R is abelian, then $\mathrm{Irr}_{1,m}(G) = \mathrm{Irr}_1(G)$, and G is solvable by [BCH]. Hence, G/R is nonabelian. Let H/R be a normal subgroup of G/R such that $|G : H| = p$ is a prime number. Let $\chi \in \mathrm{Irr}(G) - \mathrm{Irr}(G/R)$. If $\chi_H \in \mathrm{Irr}(H)$, then $(\chi_H)^G = \chi_1 + \cdots + \chi_p$, where $\chi_1 = \chi, \dots, \chi_p \in \mathrm{Irr}_1(G) - \mathrm{Irr}(G/R)$ are distinct of the same degree by reciprocity and Clifford theory. Therefore, $\chi_1, \dots, \chi_p \in \mathrm{Irr}_{1,m}(G)$ — a contradiction. Then, by Clifford theory, $\chi_H = \lambda_1 + \cdots + \lambda_p$, where $\lambda_1, \dots, \lambda_p \in \mathrm{Irr}(H)$ are distinct of the same degree. Thus, $p \mid \chi(1)$ for all $\chi \in \mathrm{Irr}(G) - \mathrm{Irr}(G/R)$. Hence, $R \leq G(p')$, and R is solvable — a contradiction.

(d) Since $\mathrm{nl}(G/\Phi(G)) = \mathrm{nl}(G)$, we obtain $R \not\leq \Phi(G)$ (in fact, by induction, $\mathrm{nl}(G/R) \leq |\mathrm{cd}_m(G/R)| < |\mathrm{cd}_m(G)|$). Let M be a maximal subgroup of G such that $R \not\leq M$. If $R < \mathrm{F}(G)$, then $M \cap \mathrm{F}(G) > \{1\}$, $\mathrm{N}_G(M \cap \mathrm{F}(G)) \geq M$ and

$\mathrm{N}_R(M \cap \mathrm{F}(G)) > \{1\}$. So $\mathrm{N}_G(M \cap \mathrm{F}(G)) = G$, G is not a monolith — a contradiction. Thus, $\mathrm{F}(G) = R$. Then, by Lemma 1(e), G has a faithful irreducible character τ such that $\tau(1) > \chi(1)$ for all $\chi \in \mathrm{Irr}(G/R)$. Thus, $|\,\mathrm{cd}_m(G/R)| \leq |\,\mathrm{cd}_m(G)| - 1 = n - 1$. Therefore, by induction, $\mathrm{nl}(G/R) \leq |\,\mathrm{cd}_m(G/R)| \leq n - 1$. Hence, $\mathrm{nl}(G) = 1 + \mathrm{nl}(G/R) \leq 1 + (n-1) = n$ — a contradiction. $\square$

Proposition 4(d) generalizes Garrison's result (see [Isa], Corollary 12.21). Proposition 4(b) generalizes [Isa], Corollary 12.6.

Remark. Let G be a solvable group such that $\mathrm{nl}(G) = |\,\mathrm{cd}(G)| = n$. Take a normal subgroup H in G such that $\mathrm{nl}(G/H) = n$, but $\mathrm{nl}(\bar{G}) < n$ for every proper epimorphic image $\bar{G}$ of G/H. It follows from Lemma 1(e) that $H < \mathrm{F}(G)$. But if $\mathrm{nl}(G) = |\,\mathrm{cd}_m(G)|$, then H is not necessarily nilpotent (see Example 2).

For a monolith G, let

$$\mu(G) = \min\{\chi(1) \,|\, \chi \in \mathrm{Irr}(G), \chi \text{ is faithful}\}.$$

A subset X of $\mathrm{Irr}_{1,m}(G)$ is said to be fundamental if whenever $\chi \in X$, then $X \cap \mathrm{Irr}_{1,m}(G/\ker(\chi)) = \{\chi\}$ and $\chi(1) = \mu(G/\ker(\chi))$. By construction, $|X|$ is the number of normal subgroups N of G such that G/N is a nonabelian monolith.

Let S be a set of simple groups. A tower of groups from S is said to be an S-group. We consider $\{1\}$ as an S-group.

Proposition 5. *Let X be a fundamental subset of $\mathrm{Irr}_{1,m}(G)$. If for every $\chi \in X$ there exist $H \leq G$ and $\lambda \in \mathrm{Irr}(H)$ such that $H/\ker(\lambda)$ is an S-group and $\chi = \lambda^G$, then G is an S-group.*

Proof. Let G be a counterexample of minimal order. Then G is a monolith. Let R be a minimal normal subgroup of G. By induction, G/R is an S-group and, by assumption, R is not an S-group. Let $\chi \in X \cap \mathrm{Irr}_{1,m}(G)$. By hypothesis, there exist $H \leq G$ and $\lambda \in \mathrm{Irr}(H)$ such that $H/\ker(\lambda)$ is an S-group and $\chi = \lambda^G$. Since χ is a faithful character of G of minimal degree and $(1_H)^G$ is reducible, $R \leq \ker((1_H)^G) \leq H$. Since $H/\ker(\lambda)$ is an S-group but R is not an S-group, it follows that $R \leq \ker(\lambda)$. Then $\{1\} < R \leq \ker(\lambda^G) = \ker(\chi) = \{1\}$ — a contradiction. $\square$

Of course, this proof is a small modification of one of two known proofs of Taketa's Theorem. Proposition 5 generalizes [Ber1], Theorem 2.

Conjecture 3. If G is as in Proposition 4(c), then $\mathrm{dl}(G) \leq 3$.[1]

Note that $|\,\mathrm{cd}_m(G)|$ is not bounded for such groups G.

Conjecture 4. If $\mathrm{cd}_m(G)$ is a chain under divisibility, then G has an ordered Sylow tower.

Conjecture 5. If every monolithic character of G is monomial, then $\mathrm{dl}(G) \leq |\,\mathrm{cd}_m(G)|$.

Conjecture 6. There exists a constant c such that $|\,\mathrm{cd}_m(G)| \leq c \cdot \mathrm{dl}(G)$.

Conjecture 7. Let G be nonsolvable. If only two characters in $\mathrm{Irr}_{1,m}(G)$ have equal degrees, then $G = L \times H$, where $L \cong \mathrm{L}_2(5)$ or $\mathrm{L}_2(7)$, $\mathrm{cd}_m(L) \bigcap \mathrm{cd}_m(H) = \{1\}$ and all characters in $\mathrm{Irr}_{1,m}(H)$ have distinct degrees.

[1] As proved by L. S. Kazarin and the author, $\mathrm{nl}(G) \leq 3$.

If G is nonsolvable and only two characters in $\mathrm{Irr}_1(G)$ have equal degrees, then $G \in \{\mathrm{L}_2(5),\ \mathrm{L}_2(7)\}$ [BK].

For further information on monolithic characters see [BZ1], Chapter 30 and [BZ2].

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