PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 3, March 1997, Pages 731–738
S 0002-9939(97)03815-X

COMMENSURATORS OF PARABOLIC SUBGROUPS OF COXETER GROUPS

LUIS PARIS

(Communicated by Ronald M. Solomon)

Abstract. Let (W, S) be a Coxeter system, and let X be a subset of S. The subgroup of W generated by X is denoted by W_X and is called a parabolic subgroup. We give the precise definition of the commensurator of a subgroup in a group. In particular, the commensurator of W_X in W is the subgroup of w in W such that $wW_Xw^{-1} \cap W_X$ has finite index in both W_X and wW_Xw^{-1}. The subgroup W_X can be decomposed in the form $W_X = W_{X^0} \cdot W_{X^\infty} \simeq W_{X^0} \times W_{X^\infty}$ where W_{X^0} is finite and all the irreducible components of W_{X^∞} are infinite. Let Y^∞ be the set of t in S such that $m_{s,t} = 2$ for all $s \in X^\infty$. We prove that the commensurator of W_X is $W_{Y^\infty} \cdot W_{X^\infty} \simeq W_{Y^\infty} \times W_{X^\infty}$. In particular, the commensurator of a parabolic subgroup is a parabolic subgroup, and W_X is its own commensurator if and only if $X^0 = Y^\infty$.

1. Introduction

Let S be a finite set. A *Coxeter matrix* over S is a matrix $M = (m_{s,t})_{s,t\in S}$ indexed by the elements of S and satisfying

(a) $m_{s,s} = 1$ if $s \in S$,
(b) $m_{s,t} = m_{t,s} \in \{2, 3, 4, \dots, +\infty\}$ if $s, t \in S$ and $s \neq t$.

A Coxeter matrix $M = (m_{s,t})_{s,t\in S}$ is usually represented by its *Coxeter graph* Γ. This is defined by the following data.

(a) S is the set of vertices of Γ.
(b) Two vertices $s, t \in S$ are joined by an edge if $m_{s,t} \geq 3$.
(c) The edge joining two vertices $s, t \in S$ is labeled by $m_{s,t}$ if $m_{s,t} \geq 4$.

The *Coxeter system* associated with M (or with Γ) is the pair (W, S) where W is the group having the presentation

$$W = \langle S \mid (st)^{m_{s,t}} = 1 \text{ if } m_{s,t} < +\infty \rangle \ .$$

The group W is called the *Coxeter group* associated with M. Given $X \subseteq S$, we write

$$M_X = (m_{s,t})_{s,t\in X},$$

$$\Gamma_X \text{ the Coxeter graph which represents } M_X,$$

$$W_X \text{ the subgroup of } W \text{ generated by } X.$$

Received by the editors October 17, 1995.

1991 *Mathematics Subject Classification.* Primary 20F55.

The pair (W_X, X) is the Coxeter system associated with M_X (see [Bo, Ch. IV, §1, n^o 8]). The group W_X is called a *parabolic subgroup* of the Coxeter system (W, S). We assume that the reader is familiar with the theory of Coxeter groups. We refer to [Bo] and [Hu] for general expositions on the subject.

For a group G and for a subgroup H of G, we denote by $Z(G)$ the center of G, by $Z_G(H)$ the centralizer of H in G, by $N_G(H)$ the normalizer of H in G, and by $C_G(H)$ the commensurator of H in G. Recall that this is defined by

$$C_G(H) = \{g \in G \ ; \ H \cap (gHg^{-1}) \ \text{ has finite index in both } H \text{ and } gHg^{-1}\} \ .$$

Commensurators play an important role in representation theory, especially in the study of induced representations. For example, if a subgroup H of G is its own commensurator, then any finite dimensional irreducible representation of H induces an irreducible representation of G (see [Ma]). If $\mathbf{K}$ is an infinite field and P is a parabolic subgroup of $\mathrm{GL}(n, \mathbf{K})$, then P is its own commensurator (see [BH]). A similar result is obviously not true for Coxeter groups. Indeed, the commensurator of a finite parabolic subgroup is the whole group W. However, we prove in this paper that the commensurator of a parabolic subgroup is always a parabolic subgroup (Corollary 2.2), and we give a criterion which decides whether a parabolic subgroup is its own commensurator (Corollary 2.3).

The goal of this paper is to determine the commensurator of a parabolic subgroup W_X of a Coxeter system. This subgroup can be decomposed in the form $W_X = W_{X^0} \cdot W_{X^\infty} \simeq W_{X^0} \times W_{X^\infty}$, where W_{X^0} is finite and all the irreducible components of W_{X^∞} are infinite. In a first step (Proposition 2.4), we prove that the commensurator of W_X is the normalizer of W_{X^∞}. In a second step (Proposition 2.5), we prove that the normalizer of W_{X^∞} is $QZ_W(W_{X^\infty}) \cdot W_{X^\infty}$, where

$$QZ_W(W_{X^\infty}) = \{w \in W \ ; \ wX^\infty w^{-1} = X^\infty\}$$

is the *quasi-centralizer* of W_{X^∞}. In a third step (Proposition 2.6), we prove that $QZ_W(W_{X^\infty})$ is W_{Y^∞}, where Y^∞ is the set of t in S such that $m_{s,t} = 2$ for all $s \in X^\infty$. Finally, from Propositions 2.4, 2.5 and 2.6, we deduce the following expression of the commensurator of W_X (Theorem 2.1):

$$C_W(W_X) = W_{Y^\infty} \cdot W_{X^\infty} \simeq W_{Y^\infty} \times W_{X^\infty} \ .$$

We state our results precisely in Section 2, and we prove them in Section 3.

2. STATEMENTS

From now on, we fix a Coxeter system (W, S).

Let X be a subset of S. Let $\Gamma_1, \dots, \Gamma_n$ be the connected components of Γ_X and, for $i \in \{1, \dots, n\}$, let X_i be the set of vertices of Γ_i. The group W_{X_i} is called an *irreducible component* of W_X. It is clear that

$$W_X = W_{X_1} \cdot \dots \cdot W_{X_n} \simeq W_{X_1} \times \dots \times W_{X_n} \ .$$

We assume that W_{X_i} is finite if $i = 1, \dots, r$, and that W_{X_i} is infinite if $i = r + 1, \dots, n$. We set

$$X^0 = X_1 \cup \dots \cup X_r \ ,$$
$$X^\infty = X_{r+1} \cup \dots \cup X_n \ .$$

Then

$$W_X = W_{X^0} \cdot W_{X^\infty} \ \simeq W_{X^0} \times W_{X^\infty} \ ,$$

the group W_{X^0} is finite, and all the irreducible components of W_{X^∞} are infinite.

Theorem 2.1. *Let X be a subset of S. Then*

$$C_W(W_X) = W_{Y^\infty} \cdot W_{X^\infty} = W_{Y^\infty \cup X^\infty} \simeq W_{Y^\infty} \times W_{X^\infty},$$

where

$$Y^\infty = \{t \in S \ ; \ m_{s,t} = 2 \ \text{ for all } s \in X^\infty\} \ .$$

Corollary 2.2. *The commensurator of a parabolic subgroup of (W, S) is a parabolic subgroup.*

Corollary 2.3. *Let X be a subset of S. Then W_X is its own commensurator if and only if X^0 is the set of $t \in S$ such that $m_{s,t} = 2$ for all $s \in X^\infty$.*

Theorem 2.1 is a direct consequence of the following Propositions 2.4, 2.5, and 2.6.

Proposition 2.4. *Let X be a subset of S. Then the commensurator of W_X in W is equal to the normalizer of W_{X^∞} in W.*

We define the *quasi-center* of (W, S) to be

$$QZ(W, S) = \{w \in W \ ; \ wSw^{-1} = S\} \ .$$

Similarly, we define the *quasi-centralizer* of a parabolic subgroup W_X of (W, S) to be

$$QZ_W(W_X) = \{w \in W \ ; \ wXw^{-1} = X\} \ .$$

Proposition 2.5. *Let X be a subset of S. Then*

$$N_W(W_X) = QZ_W(W_X) \cdot W_X \ .$$

Moreover,

$$QZ_W(W_X) \cap W_X = QZ(W_X, X) \ .$$

Proposition 2.6. *Let X be a subset of S such that all the irreducible components of W_X are infinite (i.e. $X = X^\infty$). Let*

$$Y = \{t \in S \ ; \ m_{s,t} = 2 \ \text{ for all } s \in X\} \ .$$

Then the quasi-centralizer of W_X is equal to W_Y.

Proposition 2.4 is a consequence of [So, Lemma 2]. Proposition 2.5 is stated in [Ho] for finite type Coxeter systems (see also [Kr, Ch. 3]). Moreover, its proof is quite simple. Proposition 2.6 is a consequence of [De, Prop. 5.5].

3. PROOFS

First, we state in Lemmas 3.1 and 3.2 some well-known facts that will be required later. Recall that each $w \in W$ can be written $w = s_1 \dots s_r$ where $s_i \in S$ for all $i \in \{1, \dots, r\}$. If r is as small as possible, then r is called the *length* of w and is denoted by $l(w)$.

Lemma 3.1 (Bourbaki [Bo, Ch. IV, §1, Ex. 3])**.** *Let X and X' be two subsets of S.*

(i) *Let $w \in W$. There is a unique element v of minimal length in $W_X w W_{X'}$. Moreover, each $w' \in W_X w W_{X'}$ can be written as $w' = uvu'$, where $u \in W_X$, $u' \in W_{X'}$, and $l(w') = l(u) + l(v) + l(u')$. An element v is called (X, X')-reduced if it is of minimal length in $W_X v W_{X'}$.*

(ii) *If an element v is $(X, \emptyset)$-reduced, then $l(uv) = l(u) + l(v)$ for all $u \in W_X$.*

(iii) *If an element v is $(\emptyset, X')$-reduced, then $l(vu') = l(v) + l(u')$ for all $u' \in W_{X'}$.*

(iv) *An element v is $(X, \emptyset)$-reduced if and only if $l(sv) > l(v)$ for all $s \in X$.*

(v) *An element v is $(\emptyset, X')$-reduced if and only if $l(vs') > l(v)$ for all $s' \in X'$.*

(vi) *An element v is (X, X')-reduced if and only if it is both $(X, \emptyset)$-reduced and $(\emptyset, X')$-reduced.*

Lemma 3.2 (Bourbaki [Bo, Ch. IV, §1, Ex. 22])**.** *Let w_0 be an element of W. The following statements are equivalent.*

(1) *$l(sw_0) < l(w_0)$ for all $s \in S$.*

(2) *$l(w_0 s) < l(w_0)$ for all $s \in S$.*

(3) *$l(ww_0) = l(w_0) - l(w)$ for all $w \in W$.*

(4) *$l(w_0 w) = l(w_0) - l(w)$ for all $w \in W$.*

Such an element is unique and exists if and only if W is finite. Then it is the unique element of maximal length in W. Moreover, $w_0^2 = 1$ and $w_0 S w_0 = S$.

The following proposition is the key of the proof of Proposition 2.4.

Proposition 3.3 (Solomon [So, Lemma 2])**.** *Let X and X' be two subsets of S, and let v be a (X, X')-reduced element of W. Then*

$$W_X \cap (v W_{X'} v^{-1}) = W_Y,$$

where $Y = (vX'v^{-1}) \cap X$.

Corollary 3.4. *Let X and X' be two subsets of S, and let w be an element of W. We write $w = u_0 v u_0'$, where $u_0 \in W_X$, $u_0' \in W_{X'}$, and v is (X, X')-reduced. Then*

$$W_X \cap (w W_{X'} w^{-1}) = u_0 W_Y u_0^{-1},$$

where $Y = (vX'v^{-1}) \cap X$.

Proof.

$$\begin{aligned} W_X \cap (w W_{X'} w^{-1}) &= W_X \cap (u_0 v u_0' W_{X'} u_0'^{-1} v^{-1} u_0^{-1}) \\ &= W_X \cap (u_0 v W_{X'} v^{-1} u_0^{-1}) \\ &= u_0((u_0^{-1} W_X u_0) \cap (v W_{X'} v^{-1})) u_0^{-1} \\ &= u_0 (W_X \cap (v W_{X'} v^{-1})) u_0^{-1} \\ &= u_0 W_Y u_0^{-1}\ . \end{aligned}$$

□

Proof of Proposition 2.4. Let $w \in N_W(W_{X^\infty})$. Then

$$W_{X^\infty} = w W_{X^\infty} w^{-1} \subseteq W_X \cap (w W_X w^{-1})\ ,$$

the group W_{X^∞} has finite index in W_X, and the group $w W_{X^\infty} w^{-1}$ has finite index in $w W_X w^{-1}$. Thus $W_X \cap (w W_X w^{-1})$ has finite index in both W_X and $w W_X w^{-1}$. This shows that $N_W(W_{X^\infty}) \subseteq C_W(W_X)$.

Let $w \in C_W(W_X)$. We write $w = u_0 v u_0'$, where $u_0, u_0' \in W_X$ and v is (X,X)-reduced. By Corollary 3.4,

$$W_X \cap (wW_Xw^{-1}) = u_0 W_Y u_0^{-1},$$

where $Y = (vXv^{-1}) \cap X$. Let $Y^0 = Y \cap X^0$, and let $Y^\infty = Y \cap X^\infty$. For a group G and for a subgroup H of G, we denote by $|G : H|$ the index of H in G. Then

$$\begin{aligned} |W_X : W_X \cap (wW_Xw^{-1})| &= |W_X : u_0 W_Y u_0^{-1}| = |W_X : W_Y| \\ &= |W_{X^0} : W_{Y^0}| \cdot |W_{X^\infty} : W_{Y^\infty}| \ . \end{aligned}$$

If $Y^\infty \neq X^\infty$, then, by [De, Prop. 4.2], W_{Y^∞} has infinite index in W_{X^∞}; thus $W_X \cap (wW_Xw^{-1})$ has infinite index in W_X, too. This is not the case; thus $Y^\infty = X^\infty$. Let $\Gamma_1, \dots, \Gamma_n$ be the connected components of Γ_X, and, for $i = 1, \dots, n$, let X_i be the set of vertices of Γ_i. We assume that $X^0 = X_1 \cup \cdots \cup X_r$ and that $X^\infty = X_{r+1} \cup \cdots \cup X_n$. Let $i \in \{r+1, \dots, n\}$. Then

$$v^{-1} X_i v \subseteq v^{-1} X^\infty v = v^{-1} Y^\infty v \subseteq v^{-1} Y v \subseteq X \ .$$

Thus there exists $j \in \{1, \dots, r, r+1, \dots, n\}$ such that $v^{-1} X_i v \subseteq X_j$. The group W_{X_i} is infinite and $v^{-1} W_{X_i} v \subseteq W_{X_j}$; thus W_{X_j} is infinite, and so $j \in \{r+1, \dots, n\}$. This shows that $v^{-1} X^\infty v \subseteq X^\infty$; thus $vX^\infty v^{-1} = X^\infty$; therefore $vW_{X^\infty}v^{-1} = W_{X^\infty}$. On the other hand, since $W_X = W_{X^0} \cdot W_{X^\infty} \simeq W_{X^0} \times W_{X^\infty}$, we have $uW_{X^\infty}u^{-1} = W_{X^\infty}$ for all $u \in W_X$. So,

$$wW_{X^\infty}w^{-1} = u_0 v u_0' W_{X^\infty} u_0'^{-1} v^{-1} u_0^{-1} = u_0 v W_{X^\infty} v^{-1} u_0^{-1} = u_0 W_{X^\infty} u_0^{-1} = W_{X^\infty} \ .$$

This shows that $C_W(W_X) \subseteq N_W(W_{X^\infty})$. □

Proof of Proposition 2.5. The inclusion

$$QZ_W(W_X) \cdot W_X \subseteq N_W(W_X)$$

is obvious.

Let $w \in N_W(W_X)$. We write $w = vu$, where $u \in W_X$, v is $(\emptyset, X)$-reduced, and $l(w) = l(v) + l(u)$. We have

$$wW_Xw^{-1} = vW_Xv^{-1} = W_X \ .$$

The element v is of minimal length in $vW_X = W_Xv$; thus v is also $(X, \emptyset)$-reduced. If $s \in X$, then, by Lemma 3.1,

$$\begin{aligned} & l(v) + 1 = l(vs) = l(vsv^{-1}v) = l(vsv^{-1}) + l(v) \\ & \Rightarrow l(vsv^{-1}) = 1 \\ & \Rightarrow vsv^{-1} \in W_X \cap S = X \ . \end{aligned}$$

So, $vXv^{-1} \subseteq X$; thus $vXv^{-1} = X$; therefore $v \in QZ_W(W_X)$. This shows that $N_W(W_X) \subseteq QZ_W(W_X) \cdot W_X$.

The equality

$$QZ_W(W_X) \cap W_X = QZ(W_X, X)$$

is obvious. □

Before proving Proposition 2.6, we recall some facts on root systems. Let V be a real vector space having a basis $\{e_s; s \in S\}$ in one-to-one correspondence with S. Let B be the symmetric bilinear form on V defined by

$$B(e_s, e_t) = \begin{cases} -\cos(\pi/m_{s,t}) & \text{if } m_{s,t} < +\infty, \\ -1 & \text{if } m_{s,t} = +\infty. \end{cases}$$

There is an action of W on V defined by

$$s(x) = x - 2B(x, e_s)e_s$$

if $s \in S$ and $x \in V$. This action is called the *canonical representation* of (W, S). The *root system* Φ of (W, S) is the collection of all vectors $w(e_s)$ where $w \in W$ and $s \in S$. By [Bo, Ch. V, §4, Ex. 8], every root α can be uniquely written in the form

$$\alpha = \sum_{s \in S} a_s e_s \quad (a_s \in \mathbf{R}),$$

where either all a_s are positive, or all a_s are negative. We call α *positive* and write $\alpha > 0$ if $a_s \geq 0$ for all $s \in S$. We call α *negative* and write $\alpha < 0$ if $a_s \leq 0$ for all $s \in S$.

Proposition 3.5 (Deodhar [De, Prop. 3.1])**.** *Let*

$$T = \{wsw^{-1} \ ; \ w \in W \quad \text{and } s \in S\} \ ,$$

and let Φ^+ be the set of positive roots. For $\alpha = w(e_s)$, we write $r_\alpha = wsw^{-1}$. Then the function $\Phi^+ \to T$ $(\alpha \mapsto r_\alpha)$ is well-defined and bijective.

Proposition 3.6 (Deodhar [De, Prop. 2.2])**.** *Let $w \in W$, and let $s \in S$. Then $l(ws) > l(w)$ if and only if $w(e_s) > 0$.*

For a subset X of S, we write

$$E_X = \{e_s \ ; \ s \in X\} \ .$$

The following lemma is an easy consequence of Propositions 3.5 and 3.6.

Lemma 3.7. *Let X and X' be two subsets of S, and let w be an element of W. The following statements are equivalent.*

(1) $w(E_X) = E_{X'}$.
(2) $wXw^{-1} = X'$ *and* $l(ws) > l(w)$ *for all* $s \in X$.

For $X \subseteq S$ such that W_X is finite, we denote by w_X the unique element of maximal length in W_X.

Let X be a subset of S, and let t be an element of $S \setminus X$. Let Γ_0 be the connected component of $\Gamma_{\{t\} \cup X}$ containing t, and let Y_0 be the set of vertices of Γ_0. We say that t is X-*admissible* if W_{Y_0} is finite. In that case, we write

$$c(t, X) = w_{Y_0} w_{X_0},$$

where $X_0 = Y_0 \setminus \{t\}$. It is the element of minimal length in $w_{Y_0} W_X$. In particular, $c(t, X)$ is $(\emptyset, X)$-reduced. By Lemma 3.2 and Lemma 3.7, there exists a subset X' of $\{t\} \cup X$ such that

$$c(t, X)(E_X) = E_{X'} \ .$$

If $X = X^\infty$, then t is X-admissible if and only if $m_{s,t} = 2$ for all $s \in X$. In that case, $c(t, X) = t$ and $c(t, X)(E_X) = E_X$.

Proposition 3.8 (Deodhar [De, Prop. 5.5])**.** *Let X and X' be two subsets of S, and let w be an element of W. If $w(E_X) = E_{X'}$, then there exist sequences*

$X_0 = X, X_1, \dots, X_n = X'$ *of subsets of* S,
$t_0, t_1, \dots, t_{n-1}$ *of elements of* S,

such that

(a) $t_i \in S \setminus X_i$ *and* t_i *is* X_i*-admissible* $(i = 0, 1, \dots, n-1)$,
(b) $c(t_i, X_i)(E_{X_i}) = E_{X_{i+1}}$ $(i = 0, 1, \dots, n-1)$,
(c) $w = c(t_{n-1}, X_{n-1}) \dots c(t_1, X_1)c(t_0, X_0)$.

The following Lemmas 3.9 and 3.10 are preliminary results to the proof of Proposition 2.6.

Lemma 3.9. *Let X and X' be two subsets of S, and let w be an element of W. If $wXw^{-1} = X'$, then w can be written $w = vu$, where $u \in QZ(W_X, X)$, $vXv^{-1} = X'$, and $l(vs) > l(v)$ for all $s \in X$.*

Proof. We write $w = vu$, where $u \in W_X$, v is $(\emptyset, X)$-reduced, and $l(w) = l(v) + l(u)$. We have

$$wW_Xw^{-1} = vW_Xv^{-1} = W_{X'} \ .$$

The element v is of minimal length in $vW_X = W_{X'}v$; thus v is also $(X', \emptyset)$-reduced. If $s \in X$, then, by Lemma 3.1,

$$\begin{aligned} & l(v) + 1 = l(vs) = l(vsv^{-1}v) = l(vsv^{-1}) + l(v) \\ \Rightarrow\ & l(vsv^{-1}) = 1 \\ \Rightarrow\ & vsv^{-1} \in W_{X'} \cap S = X' \ . \end{aligned}$$

So, $vXv^{-1} \subseteq X'$. Similarly, $v^{-1}X'v \subseteq X$. Thus $vXv^{-1} = X'$.

Since v is $(\emptyset, X)$-reduced, by Lemma 3.1, $l(vs) > l(v)$ for all $s \in X$.

Finally,

$$\begin{aligned} & wXw^{-1} = vuXu^{-1}v^{-1} = X' \\ \Rightarrow\ & uXu^{-1} = v^{-1}X'v = X. \end{aligned}$$

Thus $u \in QZ(W_X, X)$. ☐

Lemma 3.10 (Bourbaki [Bo, Ch. V, §4, Ex. 3])**.** *We suppose that (W, S) is irreducible.*

(i) *If W is finite, then $QZ(W, S) = \{1, w_0\}$, where w_0 is the unique element of maximal length in W.*
(ii) *If W is infinite, then $QZ(W, S) = \{1\}$.*

Proof of Proposition 2.6. The inclusion

$$W_Y \subseteq QZ_W(W_X)$$

is obvious.

Let $w \in QZ_W(W_X)$. By Lemma 3.9, w can be written $w = vu$, where $u \in QZ(W_X, X)$, $vXv^{-1} = X$, and $l(vs) > l(v)$ for all $s \in X$. Since $X = X^\infty$, by Lemma 3.10, $QZ(W_X, X) = \{1\}$; thus $u = 1$. By Lemma 3.7, $v(E_X) = E_X$. By Proposition 3.8, there exist sequences

$X = X_0, X_1, \dots, X_n = X$ of subsets of S,
$t_0, t_1, \dots, t_{n-1}$ of elements of S,

such that

(a) $t_i \in S \setminus X_i$ and t_i is X_i-admissible $(i = 0, 1, \dots, n-1)$,
(b) $c(t_i, X_i)(E_{X_i}) = E_{X_{i+1}}$ $(i = 0, 1, \dots, n-1)$,
(c) $v = c(t_{n-1}, X_{n-1}) \dots c(t_1, X_1) c(t_0, X_0)$.

Since $X = X^\infty$, if $X_i = X$, then $m_{t_i,s} = 2$ for all $s \in X$ (namely, $t_i \in Y$), $c(t_i, X_i) = t_i$, and $X_{i+1} = X$ (since $t_i(E_X) = E_X$). Since $X_0 = X$, it follows that $c(t_i, X_i) = t_i \in Y$ for all $i = 0, 1, \dots, n-1$. Thus

$$w = v = t_{n-1} \dots t_1 t_0 \in W_Y \ .$$

This shows that $QZ_W(W_X) \subseteq W_Y$. □

References

[Bo] N. Bourbaki, *"Groupes et algèbres de Lie, Chapitres IV–VI"*, Hermann, Paris, 1968. MR **39:**1590

[Br] K. S. Brown, *"Buildings"*, Springer-Verlag, New York, 1989. MR **90e:**20001

[BH] M. Burger and P. de la Harpe, *Irreducible representations of discrete groups*, in preparation.

[De] V. V. Deodhar, *On the root system of a Coxeter group*, Comm. Algebra **10** (1982), 611-630. MR **83j:**20052a

[Ho] R. B. Howlett, *Normalizers of parabolic subgroups of reflection groups*, J. London Math. Soc. (2) **21** (1980), 62–80. MR **81g:**20094

[Hu] J. E. Humphreys, *"Reflection groups and Coxeter groups"*, Cambridge studies in advanced mathematics, vol. 29, Cambridge University Press, 1990. MR **92h:**20002

[Kr] D. Krammer, *"The conjugacy problem for Coxeter groups"*, Ph. D. Thesis, Utrecht, 1994.

[Ma] G. W. Mackey, *"The theory of unitary group representations"*, The University of Chicago Press, 1976. MR **53:**686

[So] L. Solomon, *A Mackey formula in the group ring of a Coxeter group*, J. Algebra **41** (1976), 255–264. MR **56:**3104

Laboratoire de Topologie, Département de Mathématiques, Université de Bourgogne, U.M.R. 5584, B.P. 138, 21004 Dijon Cedex, France

E-mail address: `lparis@satie.u-bourgogne.fr`