

## CONJUGACY CLASSES OF SYMMETRIES IN ORTHOGONAL GROUPS

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ABSTRACT. The number of conjugacy classes of symmetries in the integral orthogonal group of an indefinite  $\mathbb{Z}$ -lattice is determined. The results are applied to the extended Bianchi groups.

### 1. INTRODUCTION

Let  $V$  be a regular quadratic space, of finite dimension  $n \geq 4$ , over the rational field  $\mathbb{Q}$  with quadratic form  $Q : V \rightarrow \mathbb{Q}$  and associated bilinear form  $2f(x, y) = Q(x + y) - Q(x) - Q(y)$ . Let  $L$  be a  $\mathbb{Z}$ -lattice on  $V$ , let  $O(L)$  be the orthogonal group of  $L$ , and let  $O'(L)$  be its spinor kernel. Then both  $O(L)$  and  $O'(L)$  act on the symmetries

$$\Psi(x) : y \rightarrow y - 2f(x, y)Q(x)^{-1}x, \quad x, y \in L,$$

in  $O(L)$  by conjugation, with  $\phi\Psi(x)\phi^{-1} = \Psi(\phi(x))$ . We study the number of conjugacy classes under both actions, and then apply the results to the extended Bianchi groups and Hilbert modular groups.

The lattice  $L$  represents  $c \in \mathbb{Z}$  if there exists an  $x \in L$  with  $Q(x) = c$ . The representation is primitive if  $\mathbb{Z}x$  is a direct summand of  $L$ . Since the symmetry  $\Psi(x)$  is to be integral, only primitive representations that satisfy the extra condition  $2f(x, L) \subseteq c\mathbb{Z}$  will be considered. Let  $N(L, c)$  be the number of these primitive representations of  $c$  modulo the action of  $O(L)$ . Then  $N(L, c)$  also counts the number of conjugacy classes of symmetries  $\Psi(x)$  with  $x$  primitive and  $Q(x) = c$ . Let  $N'(L, c)$  be the number of these primitive representations of  $c$  modulo the action of  $O'(L)$ , and let  $N'(L_p, c)$  be the corresponding number of primitive representations by the local lattice  $L_p$  over the  $p$ -adic integers  $\mathbb{Z}_p$ . Using Kneser's Strong Approximation Theorem to set up a bijection between the global and corresponding set of local orbits, as in the proof of Theorem 2.3 in [5], or Theorem 4.1 in [6], gives the following product formula.

**Theorem 1.1.** *Let  $L$  be a lattice on  $V$  with  $f(L, L) \subseteq \mathbb{Z}$  and discriminant  $D$ . Assume  $c \neq 0$  and the Witt index  $i_\infty(V \perp \langle -c \rangle) \geq 2$ . Then, for  $n \geq 4$ ,*

$$N'(L, c) = \prod_{p|2D} N'(L_p, c) < \infty.$$

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The number of conjugacy classes of symmetries under the action of  $O'(L)$  is then determined from the action of  $-I$  on the  $O'(L)$ -orbits. The value  $N(L, c)$  can be studied via the action of the quotient group  $O(L)/O'(L)$  on the  $O'(L)$ -orbits. These methods will be used to study the conjugacy classes of symmetries in  $O(L)$  in a special case corresponding to the Bianchi groups and Hilbert modular groups  $PSL(2, \mathcal{O}_d)$ . Related methods were used in [6] to classify the maximal non-elementary Fuchsian subgroups of the Bianchi groups up to conjugacy. In [11], Vulakh studied the conjugacy classes of reflections in the extended Bianchi group  $RB_d$ . Via the identification of  $RB_d$  with a suitable group  $O(L)$  (see [2, §11] or [10]), this is essentially the same as studying the conjugacy classes of symmetries for this  $O(L)$ . Theorem 3.5 is a generalization of one of Vulakh's results. Theorem 3.6 gives an analogue for the conjugacy classes in  $RB_d$  under the action of the Bianchi group. A Hilbert modular group analogue is simultaneously established. See also [2, §11] for an earlier special case obtained using Siegel's analytic theory of indefinite forms.

The next section contains some general results evaluating  $N'(L_p, c)$ . They can easily be modified to give a function field analogue where  $\mathbb{Z}$  is replaced by the polynomial ring  $\mathbb{F}[X]$  (with  $2 \neq 0$ ).

## 2. LOCAL SPINOR ORBITS

Local isometry invariants on  $x \in L_p$  under the action of  $O(L_p)$  are given in [7] and [9]. In general they are complicated; however, for odd  $p$ , the restriction  $f(x, L_p) \subseteq Q(x)\mathbb{Z}_p$  means that  $\mathbb{Z}_p x$  splits  $L_p$  as a rank one orthogonal summand (see [8, §92:6]). For each prime  $p$  let

$$L_p = J_1 \perp \cdots \perp J_t$$

be a Jordan splitting of  $L_p$  (see [8, §91C]), where each  $J_i$  is a  $p^{r_i}$ -modular lattice,  $r_1 < \cdots < r_t$ , and  $n_i = \text{rank } J_i$  are invariants of  $L_p$ . For odd  $p$  the discriminants  $d_i = (\det J_i)u_p^2$ , where  $u_p$  denotes the  $p$ -adic units, are also invariants. Note, for  $p$  odd,  $[SO(L_p) : O'(L_p)] = 4$  whenever some  $n_i \geq 2$ , and the  $r_i$  are not all even or not all odd (see [8, §92:5]). When all  $r_i$  have the same parity,  $[SO(L_p) : O'(L_p)] \leq 2$ . Also,  $SO(L_p) = O'(L_p)$  if and only if all  $n_i = 1$ , and all the square classes  $d_i\mathbb{Q}_p^{*2}$  are the same (in particular, the  $r_i$  all have the same parity).

If  $\Psi(x) \in O(L)$  with  $x$  primitive and  $Q(x) = c$ , then  $\mathbb{Z}_p x$  splits  $L_p$  as an orthogonal summand for each odd  $p$ . In particular, this implies that a Jordan splitting can be chosen with  $\mathbb{Z}_p x \subseteq J_i$  for some  $i$ . Hence  $cp^{-r_i} \in u_p$ , and this is therefore a necessary condition for  $N(L_p, c) > 0$ .

**Theorem 2.1.** *Let  $p$  be odd. Assume  $cp^{-r_i} \in u_p$  for some  $i$ . Then*

1.  $N'(L_p, c) = 0$  if  $n_i = 1$  and  $(\frac{c/d_i}{p}) = -1$ .
2.  $N'(L_p, c) = 2$  if  $n_i = 1$ ,  $(\frac{c/d_i}{p}) = 1$ , and  $r_j - r_i$  is odd for  $j \neq i$ .
3.  $N'(L_p, c) = 2$  if all  $n_j = 1$ ,  $(\frac{c/d_i}{p}) = 1$ , and the  $d_j\mathbb{Q}_p^{*2}$ , for  $j \neq i$ , take at most two values, and these values are not  $c\mathbb{Q}_p^{*2}$ .
4.  $N'(L_p, c) = 2$  if  $n_i = 2$ , all other  $n_j = 1$ , and the  $d_j\mathbb{Q}_p^{*2}$ , for  $j \neq i$ , and  $d_i c^{-1}\mathbb{Q}_p^{*2}$  take at most two values, and the product of any two values is not  $\epsilon\mathbb{Q}_p^{*2}$  with  $\epsilon$  a non-square unit.
5.  $N'(L_p, c) = 1$  otherwise.

*Proof.* Assume first  $n_i = 1$ . Let  $x \in L_p$  be primitive with  $Q(x) = c$ . Then we may choose  $J_i = \mathbb{Z}_p x$  and hence  $c \in d_i u_p^2$ . In this case, the representations of  $c$  by  $x$  and  $-x$  are not spinor equivalent when  $r_j - r_i$  is odd for all  $j \neq i$ . For assume  $\phi \in O'(L_p)$  with  $\phi(x) = -x$ . Then  $\Psi(x)\phi$  fixes  $x$ , and hence can be viewed as an isometry on the orthogonal complement of  $J_i$ . Therefore, by [8, §92:4],  $\Psi(x)\phi$  is a product of an odd number of symmetries  $\Psi(y)$  with each  $Q(y)p^{-r_j}$  a unit for some  $j$ . Calculating spinor norms then gives a contradiction. A similar argument holds if all  $n_j = 1$  as in case 3. Any two representations are  $SO(L_p)$ -equivalent (see [7]), and it follows that  $N'(L_p, c) \leq 2$  if some  $n_j \geq 2$ , since the orthogonal complement of  $x$  then admits isometries with all spinor norms in  $u_p \mathbb{Q}_p^{*2}$ . When all  $n_j = 1$ , it still follows that  $N'(L_p, c) \leq 2$ , because if all the  $d_j \mathbb{Q}_p^{*2}, j \neq i$ , are the same, then  $[SO(L_p) : O'(L_p)] \leq 2$ . In case 4,  $x$  and  $\theta(x)$  are not spinor equivalent for  $\theta \in SO(J_i)$  with spinor norm  $\epsilon \mathbb{Q}_p^{*2}$ ; take  $\theta(x) = -x$  if  $d_i \notin \mathbb{Q}_p^{*2}$ .

Now consider the case  $n_i \geq 3$ , or  $n_i = 2$  and  $n_j \geq 2$  for some  $j \neq i$ . If  $x$  and  $y$  both represent  $c$ , by [7] there exists  $\phi \in SO(L_p)$  with  $\phi(x) = y$ . It is easy to adjust and get  $\phi \in O'(L_p)$  using the orthogonal complement of  $x$ . The remaining cases are similar. □

**Corollary 2.2.** *The two spinor orbits in case 4 with  $d_i \notin \mathbb{Q}_p^{*2}$ , 2 or 3, are interchanged by  $-I$ . The two spinor orbits in case 2 are interchanged by any  $\phi \in SO(L_p)$  with spinor norm  $\eta p \mathbb{Q}_p^{*2}$ ,  $\eta$  a unit, and are fixed by all  $\psi \in SO(L_p)$  with spinor norm  $\eta \mathbb{Q}_p^{*2}$ .*

*Proof.* Assume  $Q(\pm x) = c$ . There exists  $\theta \in O(L_p)$  fixing  $x$  and with  $\phi\theta\Psi(x) \in O'(L_p)$ , using the orthogonal complement of  $x$ . Hence  $x$  is spinor equivalent to  $\phi(-x)$ . For the remaining part choose  $\theta \in SO(L_p)$  fixing  $x$  and with  $\psi\theta \in O'(L_p)$ . Then  $x$  is spinor equivalent to  $\psi(x)$ . □

A dyadic unimodular lattice  $L_2$  is *even* when  $Q(L_2) \subseteq 2\mathbb{Z}_2$ . Otherwise  $L_2$  is *odd* and has an orthogonal basis  $e_1, \dots, e_n$ . Then  $x = \sum_i a_i e_i \in L_2$  is *characteristic* if all coefficients  $a_i$  are units (see [5], [9]).

**Theorem 2.3.** *Let  $L_2$  be a dyadic unimodular lattice. Then*

1.  $N'(L_2, c) = 0$  when  $c \in 4\mathbb{Z}_2$ , or  $L_2$  is even with  $c$  a unit.
2.  $N'(L_2, c) = 3$  when  $L_2$  is odd and  $c \equiv \sum_i Q(e_i) \not\equiv 0, 4 \pmod{8}$ .
3.  $N'(L_2, c) = 1$  otherwise.

*Proof.* Let  $x \in L_2$  be primitive with  $Q(x) = c$ . Then  $2f(x, L_2) \subseteq c\mathbb{Z}_2$  forces  $c|2$ . When  $L_2$  is even, it is split by a hyperbolic plane  $\mathbb{Z}_2 u + \mathbb{Z}_2 v$ , and  $x$  is spinor equivalent to  $u + 2^{-1}cv$  (see [5] or [9, §2]). Now assume that  $L_2$  is odd and  $x = \sum_i a_i e_i$  is characteristic. Then  $c = Q(x) = \sum_i a_i^2 Q(e_i) \equiv \sum_i Q(e_i) \pmod{8}$ . By Hensel's lemma, this is a necessary and sufficient condition for the existence of a characteristic representation of  $c$ . When  $c \equiv 0 \pmod{4}$  the restriction  $c|2$  is violated. The result follows by strengthening the arguments in Lemmas 4.3 and 5.7 in [5], or as in Theorem 2.1 above using [9, §2]. □

**Corollary 2.4.** *When  $N'(L_2, c) = 3$ , the two orbits corresponding to the characteristic representations are interchanged by  $-I$  if  $n$  is odd, but are fixed by  $-I$  if  $n$  is even.*

*Proof.* The action of each  $\Psi(e_i)$  interchanges the two characteristic orbits (see Theorem 2.2 in [5]). □

For the final two theorems in this section we assume, as in Theorem 1.1, that  $L$  is a  $\mathbb{Z}$ -lattice with  $f(L, L) \subseteq \mathbb{Z}$  and  $n \geq 4$ , that  $c \neq 0$  and the index  $i_\infty(V \perp \langle -c \rangle) \geq 2$ . Also assume  $L_2$  is unimodular.

**Theorem 2.5.** *Necessary and sufficient conditions for  $N(L, c) > 0$  are:*

1.  $c \not\equiv 0 \pmod{4}$ , and  $2|c$  when  $L_2$  is even,
2. at each odd prime,  $cp^{-r_i} \in \mathfrak{u}_p$  for some  $i$ , and moreover, if  $n_i = 1$  then  $(\frac{c/d_i}{p}) = 1$ .

By Theorem 1 in [4], it suffices to assume  $V$  indefinite in Theorem 2.5 (instead of  $i_\infty(V \perp \langle -c \rangle) \geq 2$ ).

Let  $m$  be the number of odd primes  $p$  where  $N'(L_p, c) = 2$ .

**Theorem 2.6.** *Assume  $N(L, c) > 0$ . When  $m \geq 1$  and not only case 4 with  $d_i \in \mathbb{Q}_p^{*2}$  occurs, the number of conjugacy classes of symmetries  $\Psi(x)$  with  $x$  primitive and  $Q(x) = c$  under the action of  $O'(L)$  is  $2^{m-1}N'(L_2, c)$ . If  $m = 0$  and  $N'(L_2, c) = 3$ , the number of conjugacy classes is 2 when  $n$  is odd, and 3 when  $n$  is even.*

*Proof.* When  $m \geq 1$ , this follows since  $\Psi(x) = \Psi(-x)$ , but  $x$  and  $-x$  are in different  $O'(L)$ -orbits by Corollary 2.2. Use 2.4 when  $m = 0$ .  $\square$

The result is different if  $m \geq 1$  and only case 4 with  $d_i \in \mathbb{Q}_p^{*2}$  occurs, since  $-I$  now fixes all local  $p$ -adic orbits for  $p$  odd.

### 3. EXTENDED BIANCHI GROUPS

Let  $\mathcal{O}_d$  be the ring of integers in  $\mathbb{Q}(\sqrt{d})$ , where  $d$  is a square-free integer. It was shown in [6] that the Bianchi group  $PSL(2, \mathcal{O}_d)$ , for  $d < 0$  and  $d \equiv 2, 3 \pmod{4}$ , is isomorphic to  $O'(L)$ , where

$$L = \mathbb{Z}r \perp \mathbb{Z}s \perp (\mathbb{Z}u + \mathbb{Z}v) = B \perp H$$

is the lattice with  $Q(r) = 2$ ,  $Q(s) = -2d$ , and  $u, v$  are isotropic with  $f(u, v) = d$ . The extended Bianchi group  $B_d$ , namely, the maximal discrete extension of  $PSL(2, \mathcal{O}_d)$  in  $PSL(2, \mathbb{C})$ , is isomorphic to  $PSO(L)$ . The extended Bianchi group  $RB_d$  is  $B_d$  with the action of complex conjugation adjoined; it is isomorphic to  $PO(L)$ . The Hilbert modular group, where  $d > 0$ , is also isomorphic to  $O'(L)$ . For  $d \equiv 1 \pmod{4}$ ,  $L$  must be replaced by  $M = L + \mathbb{Z}2^{-1}(r - s)$ . These isomorphisms are related to those constructed over commutative rings in [3, §7.3B].

Let

$$J = \{x \in L \mid Q(x) \in 2d\mathbb{Z}\} = \mathbb{Z}dr \perp \mathbb{Z}s \perp H$$

and

$$K = \{x \in L \mid f(x, J) \subseteq 2d\mathbb{Z}\} = \mathbb{Z}r \perp \mathbb{Z}s \perp 2H.$$

Then  $J$  and  $K$  are sublattices of  $L$  that are invariant under the action of  $O(L)$ . Note that  $O(J) = O(L)$ . Clearly  $O(L) \subseteq O(J)$ . Conversely, let  $\phi \in O(J)$  with  $\phi(r) \in V$ . Since  $f(\phi(r), J) = f(r, J) = 2d\mathbb{Z}$ , it follows that  $\phi(r) \in K$  and  $\phi \in O(L)$ . Therefore, scaling  $J$  by  $d^{-1}$ ,  $O(L)$  is isomorphic to the orthogonal group of the integral form  $dX_1^2 - X_2^2 + X_3X_4$  used in [2].

We now study the conjugacy classes of symmetries in  $O(L)$  and  $O(M)$ . For the symmetry  $\Psi(x)$ , with  $x$  primitive in  $L$  and  $Q(x) = 2c$ , to be integral we need  $f(x, L) \subseteq c\mathbb{Z}$ , and hence  $Q(x)$  divides  $4d$ . First determine  $N'(L, 2c) = \prod_{p|2d} N'(L_p, 2c)$  for each  $Q(x) = 2c|4d$ . Then  $N(L, 2c)$  is obtained from the action

of the quotient group  $O(L)/O'(L)$  on the  $O'(L)$ -orbits. From [6],  $[O(L) : O'(L)] = 2^{t+2}$  where  $t$  is the number of distinct prime divisors of the discriminant of  $\mathbb{Q}(\sqrt{d})$ .

For  $w \in \mathbb{Z}r \perp \mathbb{Z}d^{-1}s$  and  $x \in L$ , let

$$E(u, w)(x) = x - f(u, x)w + f(w, x)u - 2^{-1}Q(w)f(u, x)u.$$

The Eichler transformation  $E(u, w)$  lies in  $O'(L)$ .

**Theorem 3.1.** *Let  $d$  be even. Then*

1.  $N'(L_2, 2c) = 0$  for  $c \equiv 0 \pmod{4}$ .
2.  $N'(L_2, 2c) = 1$  for  $c \equiv 1 + d, -3 \pmod{8}$ .
3.  $N'(L_2, 2c) = 2$  for  $c \equiv \pm 2 \pmod{8}$ .
4.  $N'(L_2, 2c) = 3$  for  $c \equiv 1, 1 - d \pmod{8}$ .

*Proof.* Let  $x = a_1r + a_2s + bu + b'v \in L_2$  with  $Q(x) = 2c$ . When  $4|c$  the condition  $f(x, L_2) \subseteq 4\mathbb{Z}_2$  forces  $x \in 2L_2$  so that  $N'(L_2, 2c) = 0$ . Assume  $c \not\equiv 0 \pmod{4}$ . If  $x \notin K_2$ , we may assume  $b$  is a unit and then  $x$  is spinor equivalent, via a suitable  $E(v, w) \in O'(L)$ , to  $bu + (bd)^{-1}cv$  for  $c$  even, or to  $r + bu + (bd)^{-1}(c - 1)v$  for  $c$  odd. The map  $\tau$  fixing  $r$  and  $s$ , and sending  $u$  to  $b^{-1}u$ , has spinor norm  $b\mathbb{Q}_2^{*2}$ . In both cases there exists  $\sigma \in SO(L)$  fixing  $x$  with spinor norm  $b\mathbb{Q}_2^{*2}$  (use Theorem 3.14 in [1] on the orthogonal complement of  $x$  when  $c$  is odd). Then  $\tau\sigma \in O'(L_2)$ , so we may assume  $b = 1$ . Thus, up to spinor equivalence, there is only one representation  $x \notin K_2$  with  $Q(x) = 2c$ .

If  $x \in K_2$ , then  $c \equiv a_1^2 - da_2^2 \equiv 1, 1 - d, \pm 2 \pmod{8}$ . Then, by modifying the previous argument, get  $b = 2$ . When  $c$  is even, a suitable  $E(v, w)$  sends  $x$  to a uniquely determined  $ar + s + 2u + 2b'v$  with  $a = 0$  or  $2$ . When  $c$  is odd,  $x$  can be sent to  $\pm r + as + 2u + 2b'v$ , with  $a = 0$  or  $1$  unique for  $x$ . The two sign choices lie in different spinor orbits by Theorem 3.14(i) in [1], since the orthogonal complement of  $x$  only allows isometries with spinor norm a unit.  $\square$

**Corollary 3.2.**  $N(L_2, 2c) = 2$  when  $c \equiv 1, 1 - d, \pm 2 \pmod{8}$ .

*Proof.* The  $x \in K_2$  are analogous to the characteristic  $x$  in the dyadic unimodular case. They determine an orbit that cannot be interchanged by  $O(L_2)$  with an orbit given by an  $x \notin K_2$ . The two characteristic orbits are interchanged by  $-I$  when  $c \equiv 1, 1 - d \pmod{8}$ .  $\square$

**Theorem 3.3.** *Let  $d \equiv 3 \pmod{4}$ . Then*

1.  $N'(L_2, 2c) = 0$  for  $c \equiv 0 \pmod{4}$ .
2.  $N'(L_2, 2c) = 1$  for  $c \equiv 2, 3 \pmod{4}$ .
3.  $N'(L_2, 2c) = 3$  for  $c \equiv 1 \pmod{4}$ .

*Proof.* Let  $x = a_1r + a_2s + b_1u + b_2v \in L_2$  be primitive with  $Q(x) = 2c$  and  $f(x, L_2) \subseteq c\mathbb{Z}_2$ . Then  $c|2, c|b_1$  and  $c|b_2$ . Hence there are no solutions when  $c \equiv 0 \pmod{4}$ , and  $x$  is spinor equivalent to  $r + s + 2u + 2bv \in K_2$  when  $c \equiv 2 \pmod{4}$ . Now assume  $c$  is a unit. Then  $u + cd^{-1}v$  is a representation of  $2c$ , and  $r + 2u + (2d)^{-1}(c - 1)v$  and  $s + 2u + (2d)^{-1}(c + d)v \in K_2$  give spinor inequivalent representations when  $c \equiv 1 \pmod{4}$ . These are distinct by [1], and are the only ones by arguments similar to those in Theorem 3.1.  $\square$

**Corollary 3.4.**  $N(L_2, 2c) = 2$  when  $c \equiv 1 \pmod{4}$ .

*Proof.* The two orbits corresponding to the  $x \in K_2$  are interchanged by  $\Psi(r - s) \in O(L_2)$ .  $\square$

For  $p$  a prime dividing  $d$ , put  $d = pq$  and choose  $a, b \in \mathbb{Z}$  with  $aq + bp = 1$ . Then  $\sigma(p) = -\Psi(u + bv)\Psi(bpr + au + bv) \in SO(L)$  has spinor norm  $p\mathbb{Q}^{*2}$ . Let  $m = m(c)$  be the number of odd primes dividing  $d$  with  $(p, c) = 1$  and  $(\frac{c}{p}) = 1$ . The following gives the number  $N(L, c)$  of conjugacy classes under  $O(L)$  of symmetries  $\Psi(x)$  with  $x$  primitive and  $Q(x) = 2c$ .

**Theorem 3.5.** *Assume that  $c > 0$  when  $d < 0$ , that  $N(M, 2c) > 0$  when  $d \equiv 1 \pmod{4}$ , and  $N(L, 2c) > 0$  when  $d \equiv 2, 3 \pmod{4}$ . Then*

1.  $N(M, 2c) = 2^{-m}N'(M, 2c) = 1$  for  $d \equiv 1 \pmod{4}$ .
2.  $N(L, 2c) = 2^{-m}N'(L, 2c) = N'(L_2, 2c)$  for  $c \not\equiv 1, 1-d \pmod{8}$  when  $d$  is even.
3.  $N(L, 2c) = 1$  and  $N'(L, 2c) = 2^m$  for  $c \equiv 2, 3 \pmod{4}$  when  $d \equiv 3 \pmod{4}$ .
4.  $N(L, 2c) = 2$  and  $N'(L, 2c) = 2^{m+3}$  for  $c \equiv -d \equiv 1 \pmod{4}$ , and for  $c \equiv 1, 1-d \pmod{8}$  when  $d$  is even.

*Proof.* When  $d \equiv 1 \pmod{4}$ ,  $M_2$  is an even unimodular lattice and hence, by Theorems 1.1 and 2.1,  $N'(M, 2c) = 2^m$ . The group  $O(M)$  is generated over  $O'(M)$  by  $\sigma(p_i), 1 \leq i \leq t, \Psi(u - v)$  and  $\Psi(u + v)$ . The result now follows since  $\sigma(p_i)$  with  $(p_i, c) = 1$  interchanges the two  $O'(M_{p_i})$ -orbits and leaves all other  $O'(M_{p_j})$ -orbits unchanged by Corollary 2.2. Each  $\sigma(p)$  corresponding to a  $p$  counted in  $m$  thus independently halves the total number of  $O'(M)$ -orbits. Thus  $N(M, 2c) = 1$ .

The argument for  $d$  even is essentially the same, with  $\sigma(p)$  having no effect on the local dyadic orbits for odd  $p$ . By Corollary 3.2 the  $O'(L_2)$ -orbits are only effected by  $O(L)$  when  $c \equiv 1, 1 - d \pmod{8}$ , and then  $-I$  reduces the number of dyadic orbits from 3 to 2.

For  $d \equiv 3 \pmod{4}$  the dyadic orbits must again be considered when  $c \equiv 1 \pmod{4}$ . Now

$$\Psi(adr + as - 2abu - 2v)\Psi(v - au) \in SO(L),$$

where  $2q = 1 - d$  and  $aq - 2b = 1$ , has spinor norm  $2\mathbb{Q}^{*2}$ , interchanges the two dyadic orbits corresponding to the representatives from  $K_2$ , and fixes the local orbits at odd primes by Corollary 2.2. □

This is essentially a refined version of Theorem 11 in [11] where only an upper bound is given for the total number of conjugacy classes of reflections in  $RB_d$ . Explicit examples, similar to those given in Theorem 11.3 of [2] and Theorem 12 of [11], can be constructed from the local information. That  $N(M, 2c) = 1$  also follows from Theorems 2 and 4 in [4].

**Theorem 3.6.** *Assume  $d \equiv 2, 3 \pmod{4}$ , that  $c > 0$  when  $d < 0$ , and  $N(L, 2c) > 0$ . When  $m \geq 1$ , the number of conjugacy classes of symmetries  $\Psi(x)$  with  $x$  primitive and  $Q(x) = 2c|4d$ , under the action of  $O'(L)$ , is  $2^{m-1}N'(L_2, 2c)$ . When  $m = 0$ , this number is  $N(L_2, 2c)$ , except for  $c \equiv -d \equiv 1 \pmod{4}$  where it is 3.*

*Proof.* The action of  $-I$  halves the number of orbits by interchanging in pairs the local  $O'(L)$ -orbits at odd primes when  $m > 0$ . Use Corollary 3.2 when  $m = 0$ . □

This essentially gives the number of conjugacy classes of reflections in  $RB_d$  under the action of  $PSL(2, O_d), d < 0$ . The corresponding result for  $O(M), d \equiv 1 \pmod{4}$ , is already covered in Theorem 2.6.

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