

A TWO-PARAMETER “BERGMAN SPACE” INEQUALITY

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ABSTRACT. For $f \in L^1([0, 1] \times [0, 1])$, define $\lambda_R \equiv \langle f, h_{(R)} \rangle$, where $h_{(R)}(x, y) = h_{(I)}(x) \cdot h_{(J)}(y)$ is a tensor product of one-parameter Haar functions. Let $1 < p \leq q < \infty$ and $q \geq 2$. We prove a sufficient condition, which is close to necessary, on double sequences of weights $\{\mu_R\}_R$ and non-negative $v \in L^1([0, 1] \times [0, 1])$, which ensures that the inequality

$$\left(\sum_R |\lambda_R|^q \mu_R \right)^{1/q} \leq \left(\int_{[0,1] \times [0,1]} |f|^p v \, dx \right)^{1/p}$$

holds for all $f \in L^1([0, 1] \times [0, 1])$. We extend our result to an inequality concerning two-parameter wavelet families.

If $I \subset [0, 1]$ is a dyadic interval, we may set

$$h_{(I)}(x) = \begin{cases} |I|^{-1/2} & \text{if } x \in I_L, \\ -|I|^{-1/2} & \text{if } x \in I_R, \\ 0 & \text{if } x \notin I, \end{cases}$$

where I_L and I_R are respectively the left and right halves of I , and $|\cdot|$ denotes Lebesgue measure. The function $h_{(I)}$ is the *Haar function* supported on I . It is well known (and easy to prove) that $\{h_{(I)}\}_I$ is an orthonormal system in $L^2([0, 1])$. If $f \in L^1[0, 1]$, we define the *Haar coefficients* λ_I of f by:

$$\begin{aligned} \lambda_I &= \langle f, h_{(I)} \rangle \\ &= \int_0^1 f(x) h_{(I)}(x) \, dx. \end{aligned}$$

The numbers λ_I are “discrete” analogues of wavelet coefficients. They measure, very crudely, how much of the frequency $\approx |I|^{-1}$ the function f has on the interval I .

It is easy to see how a single Haar coefficient is controlled by the size of f , and in particular how it is affected by small perturbations in f . It is more difficult to see how a weighted average of a (possibly infinite) collection of λ_I 's is so affected. To better understand this dependence, it is natural to consider weighted norm

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inequalities of the following form:

$$(1) \quad \left(\sum_I |\lambda_I|^q \mu_I \right)^{1/q} \leq \left(\int_0^1 |f|^p v \, dx \right)^{1/p},$$

where $\mathcal{M} = \{\mu_I\}_I$ is a sequence of non-negative numbers, and v is a non-negative function in $L^1([0, 1])$. (The exponents p and q are assumed to lie between 1 and ∞ .) One then looks for sufficient conditions on $\{\mu_I\}_I$ and v which imply that (1) holds for all $f \in L^1([0, 1])$. Naturally, one hopes that such sufficient conditions will be close to necessary.

Richard Wheeden and the author investigated this question in [WW]. They found both sufficient and necessary conditions for (1), that used an auxiliary function defined from v . Let p' be the conjugate exponent to p . They defined $\sigma \equiv v^{1-p'}$. We claim that, if (1) is true, then

$$(2) \quad \frac{(\int_I \sigma \, dx)^{1/p'} \mu_I^{1/q}}{|I|^{1/2}} \leq 1$$

must hold for all dyadic intervals I . (In particular, this implies that $\int_I \sigma$ is finite whenever $\mu_I \neq 0$.) To obtain (2), set, for $n = 1, 2, \dots$, $v_n = \max(v, 1/n)$ and $\sigma_n = v_n^{1-p'}$. Clearly, $\sigma_n \in L^\infty$, and $\int_I \sigma_n > 0$ for all I (because v is finite a.e.). If (1) holds for v , then it holds for $v_n \geq v$. Fix a dyadic interval I and set $f_n(x) = \sigma_n \cdot (\chi_{I_L}(x) - \chi_{I_R}(x))$. If we plug f_n into (1) (with v_n in place of v) and use the fact that $\sigma_n^p v_n = \sigma_n$, we obtain

$$\frac{(\int_I \sigma_n \, dx)^{1/p'} \mu_I^{1/q}}{|I|^{1/2}} \leq 1.$$

Inequality (2) follows now by monotone convergence.

The sufficient condition they proved (valid, however, only for a limited range of p 's and q 's) is just a bit more restrictive. For any non-negative real number η , and any dyadic interval I , they defined

$$\sigma(I, \eta) = \int_I \sigma(x) \log^\eta(e + \sigma(x)/\sigma_I) \, dx,$$

where σ_I denotes σ 's average over I . They proved ([WW], Theorem 1):

Theorem 1. *Let v , σ , and \mathcal{M} be as above. Let $1 < p \leq q < \infty$ and $q \geq 2$. Suppose that $\eta > p'/2$. There is a positive constant $C = C(p, q, \eta)$ such that if*

$$\frac{\sigma(I, \eta)^{1/p'} \mu_I^{1/q}}{|I|^{1/q}} \leq C$$

for all dyadic intervals I , then (1) holds.

The authors also proved an analogous result for wavelet expansions ([WW], Theorem 2).

Inequality (1) has a natural reformulation in a two-parameter setting. If $R = I \times J$ is a Cartesian product of dyadic intervals, we can set $h_{(R)}(x, y) = h_{(I)}(x) \cdot h_{(J)}(y)$, the tensor product of two ‘‘one-parameter’’ Haar functions, and define $\lambda_R \equiv \langle f, h_{(R)} \rangle$ for $f \in L^1([0, 1] \times [0, 1])$. The coefficients λ_R have a meaning analogous to that of the λ_I 's, except that now the ‘‘frequencies’’ are two-dimensional vectors and are supported on rectangles. We want strong (i.e., close to necessary)

conditions on double sequences of non-negative weights $\mathcal{N} = \{\mu_R\}_R$ and non-negative functions $v \in L^1([0, 1] \times [0, 1])$ which ensure that

$$(3) \quad \left(\sum_R |\lambda_R|^q \mu_R \right)^{1/q} \leq \left(\int_{[0,1] \times [0,1]} |f|^p v \, dx \right)^{1/p}$$

holds for all $f \in L^1([0, 1] \times [0, 1])$. (As before, we assume that p and q lie strictly between 1 and ∞ .)

Set $\sigma = v^{1-p'}$. An argument like the one given above shows that, in order for (3) to hold, we must have

$$\frac{\left(\int_R \sigma \, dx \, dy \right)^{1/p'} \mu_R^{1/q}}{|R|^{1/2}} \leq 1$$

for all double-dyadic rectangles R . This fact lends superficial plausibility to the idea that a sufficient condition for (3) should have a form like that given in Theorem 1, at least for the same range of p 's and q 's.

For R a double-dyadic rectangle, σ a non-negative measurable function, and η a non-negative number, define

$$\tilde{\sigma}(R, \eta) \equiv \int_R \sigma(x, y) \log^\eta(e + \sigma/\sigma_R) \, dx \, dy,$$

where σ_R is σ 's average over R . (We are calling this new functional ‘ $\tilde{\sigma}(R, \eta)$ ’, instead of ‘ $\sigma(R, \eta)$ ’, because we still have need of the latter.) Now we can state our theorem.

Theorem 2. *Let v be a non-negative function in $L^1([0, 1] \times [0, 1])$. Let $\mathcal{N} = \{\mu_R\}_R$ be a sequence of non-negative numbers indexed over the double-dyadic rectangles in $[0, 1] \times [0, 1]$. Let $1 < p \leq q < \infty$ and $q \geq 2$. Set $\sigma = v^{1-p'}$. For every $\eta > p'$, there is a positive constant $C = C(p, q, \eta)$ such that if*

$$(4) \quad \frac{\tilde{\sigma}(R, \eta)^{1/p'} \mu_R^{1/q}}{|R|^{1/2}} \leq C$$

for all R , then (3) holds for all $f \in L^1([0, 1] \times [0, 1])$.

Proof. Up to a point, the proof follows that of Theorem 1 in [WW]. Let \mathcal{R} denote the family of all double-dyadic rectangles in $[0, 1] \times [0, 1]$. For $g: \mathcal{R} \mapsto \mathbf{R}$ a finitely supported function, set

$$Tg(x, y) = \sum_R g(R) \mu_R h_{(R)}(x, y).$$

The sum is well-defined because g has finite support. By duality, inequality (3) will hold if

$$(5) \quad \left(\int_{[0,1] \times [0,1]} |Tg|^{p'} \sigma \, dx \, dy \right)^{1/p'} \leq \left(\sum_R |g(R)|^{q'} \mu_R \right)^{1/q'}$$

for all such g .

As in the proof of Theorem 1 in [WW], the proof that (4) implies (5) breaks into two cases: $p' < 2$ and $p' \geq 2$. The first case follows from a fairly straightforward iteration of one-dimensional results, and we will treat it first.

Fix $y \in [0, 1]$. Set $R_y(x) = Tg(x, y)$ and $\nu_y(x) = \sigma(x, y)$. We may write $R_y(x)$ as a linear combination of one-dimensional Haar functions:

$$\begin{aligned} R_y(x) &= \sum_I \left(\sum_{J \ni y} g(I \times J) \mu_{I \times J} h_{(J)}(y) \right) \cdot h_{(I)}(x) \\ &= \sum_I \gamma_I^{(y)} h_{(I)}(x). \end{aligned}$$

Let $\tau = \eta/2 > p'/2$. Following the argument in [WW] (Theorem 1), we see that

$$\int_0^1 |R_y(x)|^{p'} \nu_y(x) dx \leq C \sum_I \frac{|\gamma_I^{(y)}|^{p'}}{|I|^{p'/2}} \nu_y(I, \tau),$$

where the constant C depends on p' and τ .

By Young's Inequality, we can dominate $\nu_y(I, \tau)$ by an integral of the form

$$(6) \quad \int_I \nu_y(x) \psi(I, x, y) dx,$$

where $\psi(I, x, y)$ satisfies

$$(7) \quad \frac{1}{|I|} \int_I \exp(\psi(I, x, y)^{1/\tau}) dx \leq 6.$$

(See [W], Theorem 2.1.) Clearly, the functions $\psi(I, x, y)$ can be taken to be measurable in (x, y) .

Call the quantity in (6) $\Omega^I(y)$. We have shown that, for every y ,

$$\int_0^1 |R_y(x)|^{p'} \nu_y(x) dx \leq C(\tau, p') \sum_I \frac{|\gamma_I^{(y)}|^{p'}}{|I|^{p'/2}} \Omega^I(y).$$

Let us now take *one term* from the preceding right-hand sum and integrate it in y .

Recalling that $\gamma_I^{(y)}$ is itself a sum of Haar functions (in $y!$), a verbatim repetition of the argument from [WW] shows that

$$\begin{aligned} \int_0^1 \frac{|\gamma_I^{(y)}|^{p'}}{|I|^{p'/2}} \Omega^I(y) dy &\leq \frac{C(\tau, p')}{|I|^{p'/2}} \sum_J \frac{|g(I \times J)|^{p'} \mu_R^{p'}}{|J|^{p'/2}} \Omega^I(J, \tau) \\ (8) \quad &\leq \frac{C'(\tau, p')}{|I|^{p'/2}} \sum_J \frac{|g(I \times J)|^{p'} \mu_R^{p'}}{|J|^{p'/2}} \\ &\quad \times \int_{I \times J} \sigma(x, y) \psi(I, x, y) \phi(I, J, y) dx dy, \end{aligned}$$

where each function $\phi(I, J, y)$ satisfies

$$(9) \quad \frac{1}{|J|} \int_J \exp(\phi(I, J, y)^{1/\tau}) dy \leq 6.$$

Plugging (8) back into our sum yields:

$$\begin{aligned} &\int_{[0,1] \times [0,1]} |Tg|^{p'} \sigma dx dy \\ &\leq C''(\tau, p') \sum_{R=I \times J} \frac{|g(I \times J)|^{p'} \mu_R^{p'}}{|R|^{p'/2}} \int_{I \times J} \sigma(x, y) \psi(I, x, y) \phi(I, J, y) dx dy. \end{aligned}$$

Since $\psi(I, x, y)$ and $\phi(I, J, y)$ satisfy respectively (7) and (9) uniformly on the (respectively) x - and y -slices of $R = I \times J$, the Cauchy-Schwarz inequality implies that their product, $\rho(I, J, x, y)$, must satisfy

$$\frac{1}{|R|} \int_I \exp(\rho(I, J, x, y)^{1/2\tau}) \, dx \, dy \leq 6.$$

Now Young’s Inequality (again) implies that

$$\int_{I \times J} \sigma(x, y) \psi(I, x, y) \phi(I, J, y) \, dx \, dy \leq C \tilde{\sigma}(I \times J, 2\tau).$$

Thus,

$$\begin{aligned} \int_{[0,1] \times [0,1]} |Tg|^{p'} \sigma \, dx \, dy &\leq C''(\tau, p') \sum_{R=I \times J} \frac{|g(I \times J)|^{p'} \mu_R^{p'}}{|R|^{p'/2}} \tilde{\sigma}(R, 2\tau) \\ &= C''(\tau, p') \sum_R \frac{|g(R)|^{p'} \mu_R^{p'}}{|R|^{p'/2}} \tilde{\sigma}(R, \eta), \end{aligned}$$

since $\eta = 2\tau$. The hypothesis of Theorem 2 implies that the last line is bounded by

$$C \sum_R |g(R)|^{p'} \mu_R^{p'/q'} \leq C \left(\sum_R |g(R)|^{q'} \mu_R \right)^{p'/q'},$$

where the right-hand side follows because $p' \geq q'$. This proves the first case.

The proof of the second ($p' \geq 2$) case which we will give does *not* follow from a simple iteration of the corresponding one-dimensional argument. The one-parameter proof (Theorem 1 from [WW]) used some fairly recent results about one-parameter weighted Littlewood-Paley theory. Unfortunately, we do not yet have proofs of the analogous two-parameter Littlewood-Paley results, except in the case where we are mapping $L^2 \mapsto L^2$. Nor do we prove such inequalities here. Instead, we prove the “Bergman space” inequality (for $p' \geq 2$) using *only* a weighted $L^2 \mapsto L^2$ result. Our argument also works in the one-parameter setting, and in that case it yields a much simpler proof than the one given in [WW].

We will need the following result from [W] (Theorem 2.1): *Let \mathcal{F} be a finite family of double-dyadic rectangles and let $\tau > 2$. There is a constant C_τ such that, for all functions*

$$f(x, y) = \sum_{R \in \mathcal{F}} \lambda_R h_{(R)}(x, y)$$

and all non-negative $\nu \in L^1([0, 1] \times [0, 1])$,

$$(10) \quad \int_{[0,1] \times [0,1]} |f|^2 \nu \, dx \, dy \leq C_\tau \sum_{R \in \mathcal{F}} \frac{|\lambda_R|^2}{|R|} \tilde{\nu}(R, \tau).$$

Let us now take $g: \mathcal{R} \mapsto \mathbf{R}$ to be a finitely supported function. We wish to prove (5). Let ρ be the dual exponent to $p'/2$ (which, recall, is ≥ 1). There exists a non-negative $\phi \in L^\rho(\sigma)$ with $\|\phi\|_{L^\rho(\sigma)} = 1$ such that

$$(11) \quad \int_{[0,1] \times [0,1]} |Tg|^{p'} \sigma \, dx \, dy = \left(\int_{[0,1] \times [0,1]} |Tg|^2 \phi \sigma \, dx \, dy \right)^{p'/2}.$$

We set $\aleph = \phi \cdot \sigma$, and use (10) to bound the right-hand side of (11) by

$$\left(C_\tau \sum_R \frac{|g(R)\mu_R|^2}{|R|} \tilde{\aleph}(R, \tau) \right)^{p'/2},$$

for $\tau = 2\eta/p' > 2$.

By Young's Inequality, the quantity $\tilde{\aleph}(R, \tau)$ can be dominated by

$$(12) \quad C \int_R \aleph(x, y) \psi(R, x, y) dx dy = C \int_R \phi(x, y) \psi(R, x, y) \sigma dx dy,$$

where $\psi(R, x, y)$ satisfies

$$\frac{1}{|R|} \int_R \exp(\psi(R, x, y)^{1/\tau}) dx dy \leq 6.$$

Because of Hölder's Inequality (and the fact that $\|\phi\|_{L^\rho(\sigma)} \leq 1$), the right-hand side of (12) is less than or equal to

$$(13) \quad C \left(\int_R \psi(R, x, y)^{p'/2} \sigma dx dy \right)^{2/p'}.$$

The function $\beta(x, y) = \psi(R, x, y)^{p'/2}$ satisfies

$$\frac{1}{|R|} \int_R \exp(\beta(x, y)^{2/(p'\tau)}) dx dy \leq 6.$$

Therefore, Young's Inequality implies that (13) is less than or equal to $C\tilde{\sigma}(R, p'\tau/2)^{2/p'}$. When we substitute this back into (11), we get:

$$(14) \quad \int_{[0,1] \times [0,1]} |Tg|^{p'} \sigma dx dy \leq C_\tau \left(\sum_R \frac{|g(R)\mu_R|^2}{|R|} \tilde{\sigma}(R, p'\tau/2)^{2/p'} \right)^{p'/2}.$$

Since $\eta = p'\tau/2$, the hypothesis of Theorem 2 implies that the right-hand side of (14) is dominated by

$$C'_\eta \left(\sum_R |g(R)|^2 \mu_R^{2/q'} \right)^{p'/2},$$

which in turn is less than or equal to

$$C'_\eta \left(\sum_R |g(R)|^{q'} \mu_R \right)^{p'/q'},$$

because $q' \leq 2$. Theorem 2 is proved.

Theorem 2 has an immediate corollary relating to wavelet expansions. Let ψ be a smooth function supported in $[-1, 2]$, and which also satisfies $\int_{\mathbf{R}} \psi = 0$. If $I \subset \mathbf{R}$ is a dyadic interval with left endpoint x^* , we set $\psi_{(I)}(x) = |I|^{-1/2} \psi((x - x^*)/\ell(I))$, where $\ell(I)$ denotes the length of I . (If ψ is chosen cleverly, the family $\mathcal{G} = \{\psi_{(I)}\}_I$, called a *wavelet system*, is an orthonormal basis for $L^2(\mathbf{R})$.)

We wish to consider a two-parameter version of \mathcal{G} . For $R = I \times J \subset \mathbf{R}^2$, a double-dyadic rectangle, set

$$\Psi_{(R)}(x, y) = \psi_{(I)}(x) \cdot \psi_{(J)}(y).$$

Set $\mathcal{DG} = \{\Psi_{(R)}\}_R$. Just as with \mathcal{G} , the right choice of ψ makes \mathcal{DG} into an orthonormal basis—this time for $L^2(\mathbf{R}^2)$.

For $f \in L^1_{loc}(\mathbf{R}^2)$, set $\lambda_R^* = \langle f, \Psi_{(R)} \rangle$. This coefficient has the same significance as λ_R , but it is, as a rule, a more reliable and useful measure of f 's local spectrum than λ_R .

For the reasons given in the introduction, it is natural to ask for what non-negative weights $v \in L^1_{loc}(\mathbf{R}^2)$ and double sequences of weights $\{\mu_R\}$ the inequality

$$(15) \quad \left(\sum_R |\lambda_R^*|^q \mu_R \right)^{1/q} \leq \left(\int_R |f|^p v \, dx \right)^{1/p}$$

holds for all $f \in L^1_{loc}(\mathbf{R}^2)$. Fortunately, one answer (along with its proof) is virtually the same as that for the double-dyadic Haar coefficients, at least if we consider the same range of p 's and q 's. For I a dyadic interval, let \tilde{I} denote I 's triple. If $R = I \times J$, we set $\tilde{R} = \tilde{I} \times \tilde{J}$. In [W] (Theorem 2.2) it is proved that if $\mathcal{DF} \subset \mathcal{DG}$ is any finite family and

$$g(x, y) = \sum_{R \in \mathcal{DF}} \gamma_R \Psi_{(R)}(x, y),$$

where the γ_R 's are real numbers, then, for any $\tau > 2$ and any weight $\sigma \in L^1_{loc}(\mathbf{R}^2)$,

$$(16) \quad \int_{\mathbf{R}^2} |g|^2 \sigma \, dx \leq C_{\tau, \psi} \sum_R \frac{|\gamma_R|^2}{|R|} \tilde{\sigma}(\tilde{R}, \tau).$$

This fact is the key to the “hard” ($p' \geq 2$) case of the following

Theorem 3. *Let \mathcal{DG} be a two-parameter wavelet system, given as above. Let v be a non-negative function in $L^1_{loc}(\mathbf{R}^2)$. Let $\mathcal{N} = \{\mu_R\}_R$ be a sequence of non-negative numbers indexed over the double-dyadic rectangles in \mathbf{R}^2 . Let $1 < p \leq q < \infty$ and $q \geq 2$. Set $\sigma = v^{1-p'}$. For every $\eta > p'$, there is a positive constant $C = C(p, q, \eta, \psi)$ such that if*

$$(17) \quad \frac{\tilde{\sigma}(\tilde{R}, \eta)^{1/p'} \mu_R^{1/q}}{|R|^{1/2}} \leq C$$

for all R , then (15) holds for all $f \in L^1_{loc}(\mathbf{R}^2)$.

Proof. Let $g: \mathcal{DG} \mapsto \mathbf{R}$ have finite support and define

$$\tilde{T}g(x, y) = \sum_{R \in \mathcal{DG}} g(R) \mu_R \Psi_{(R)}(x, y),$$

analogous to Tg defined earlier. We wish to establish the inequality

$$\left(\int_{\mathbf{R}^2} |\tilde{T}g|^{p'} \sigma \, dx \, dy \right)^{1/p'} \leq \left(\sum_R |g(R)|^q \mu_R \right)^{1/q'}$$

under the assumption that (17) holds.

As before, the argument breaks into two cases; and, just as before, the ‘ $p' \leq 2$ ’ case follows from a direct iteration of one-parameter results. Note that, for fixed y ,

$$\begin{aligned} \tilde{T}g(x, y) &= \sum_I \left(\sum_{y \in \tilde{J}} g(\tilde{I} \times \tilde{J}) \mu_R \psi_{(J)}(y) \right) \cdot \psi_{(I)}(x) \\ &= \sum_I \gamma_I^y \psi_{(I)}(x). \end{aligned}$$

Let $\tau = \eta/2 > p'/2$. By the proof of Theorem 2 (Subcase 1': $2 < p < \infty$) in [WW], we see that, for every y ,

$$\int_{\mathbf{R}} |\tilde{T}g|^{p'} \sigma(x, y) dx \leq C_{\tau, \psi} \sum_I \frac{|\gamma_I^y|^{p'}}{|I|^{p'/2}} \nu_y(\tilde{I}, \tau),$$

where, as before, we have set $\nu_y(x) = \sigma(x, y)$. But this is precisely the same sort of expression we got in the earlier Haar function case! If we integrate in y now, and combine our Young’s Inequality argument with the result from Theorem 2 in [WW] (which is the “smooth” analogue of Theorem 1), we get the ‘ $p' \leq 2$ ’ result here as well.

The other case is just as easy. Let ρ be the dual exponent to $p'/2 \geq 1$. There is a non-negative $\phi \in L^\rho(\sigma)$ such that $\|\phi\|_{L^\rho(\sigma)} = 1$ and

$$\int_{\mathbf{R}^2} |\tilde{T}g|^{p'} \sigma dx dy = \left(\int_{\mathbf{R}^2} |\tilde{T}g|^2 \phi \sigma dx dy \right)^{p'/2}.$$

By inequality (16) above, the right-hand integral is dominated by

$$C_{\eta, \psi} \sum_R \frac{|g(R) \mu_R|^2}{|R|} \tilde{\aleph}(\tilde{R}, \eta),$$

where we set $\aleph = \sigma \cdot \phi$. Once again, this is exactly what we got in the Haar function case, and the proof concludes the same way. QED

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