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Q.U.P. AND PALEY-WIENER PROPERTIES OF UNIMODULAR, ESPECIALLY NILPOTENT, LIE GROUPS

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Abstract. We give a new proof of a weak Paley-Wiener theorem for nilpotent Lie groups due to Lipsman and Rosenberg and we introduce a general notion of Q.U.P for any unimodular locally compact group.

0. Introduction

0.0. Let G be a locally compact group equipped with left Haar measure m. $\widehat{G}$ will denote the dual of G (i.e. a maximal set of pairwise disjoint inequivalent continuous irreducible unitary representations of G). The Fourier transform $\hat{f}$ of a $L^1(G)$ function f is defined by:

$$\hat{f}(\pi) = \pi(f) = \int_G f(x)\pi(x)\,dm(x), \qquad \pi \in \widehat{G}.$$

If f, g are measurable functions on G, we define their convolution $f * g$ by

$$f * g(x) = \int_G f(y)g(y^{-1}x)\,dm(y)$$

whenever the integral exists.

For a function f on G, let

$$f^*(x) = \overline{f}(x^{-1}), \qquad x \in G,$$

and

$${}_xf(y) = f(x^{-1}y), \quad f_x(y) = f(yx^{-1}), \qquad x, y \in G.$$

For f in $L^1(G)$, let $A_f = \{x \in G; f(x) \neq 0\}$ and $B_f = \{\pi \in \widehat{G}; \hat{f}(\pi) \neq 0\}$.

In 1973, Matolcsi and Szücs [Mat,Szu] showed that if G is a locally compact abelian group, then with notation as above, $m(A_f)\hat{m}(B_f) < 1$ implies $f = 0$ almost everywhere (here $\hat{m}$ is the Haar measure on the dual group $\widehat{G}$, normalized so that the Plancherel identity is valid). In a 1974 preprint (which was published much later), Benedicks [Ben] showed that for $f \in L^1(\mathbb{R}^n)$, $m(A_f)\hat{m}(B_f) < \infty$ implies $f = 0$ almost everywhere, and in 1977 Amrein and Berthier [Amr,Ber] reached the same conclusion using Hilbert space techniques.

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0.1. Now let G be a unimodular group of type I. Then there exists a measure $\hat{m}$ on $\widehat{G}$ such that

$$L^2(G) \simeq L^2(\widehat{G}) = \int_{\widehat{G}}^{\oplus} \mathrm{End}_2(\mathcal{H}_\pi)\, d\hat{m}(\pi),$$

i.e. the Fourier transform $f \mapsto \hat{f}$ may be extended to $L^2(G)$ to furnish an isometric isomorphism between $L^2(G)$ and the space $L^2(\widehat{G})$ of the measurable fields $h : \pi \mapsto h(\pi)$, with $h(\pi)$ in $\mathrm{End}_2(\mathcal{H}_\pi)$ and $\int_{\widehat{G}} \|h(\pi)\|_2^2\, d\hat{m}(\pi) < \infty$.

Here $\mathrm{End}_2(\mathcal{H})$ denotes the space of the Hilbert-Schmidt operators on the Hilbert space $\mathcal{H}$ and $\|T\|_2$ is the Hilbert-Schmidt norm

$$\|T\|_2^2 = \sum_k \|T\phi_k\|_{\mathcal{H}}^2, \qquad T \in \mathrm{End}_2(\mathcal{H}),$$

where $\{\phi_k\}$ is any orthonormal basis of $\mathcal{H}$.

If $E \subset \widehat{G}$ is a measurable subset, we define $\mu(E)$ according to Hogan (see [Hog]) by

$$\mu(E) = \int_E (\dim \mathcal{H}_\pi)\, d\hat{m}(\pi).$$

In [Hog], a unimodular locally compact group G of type I is said to satisfy the qualitative uncertainty principle (Q.U.P.) if, for each $f \in L^2(G)$,

$$m(A_f) < m(G) \text{ and } \mu(B_f) < \mu(\widehat{G}) \Rightarrow f = 0 \quad m\text{-a.e.}$$

It is shown in [Hog], Theorem 2.2, that any unimodular type I group with noncompact identity component satisfies the Q.U.P.

0.2. The condition $\mu(B_f) < \infty$ implies that for almost all $\pi \in B_f$, $\dim(\mathcal{H}_\pi) < \infty$, which is rather restrictive, and that

$$\int_{\widehat{G}} \mathrm{rank}(\pi(f))\, d\hat{m}(\pi) < \infty, \tag{0.2.1}$$

where $\mathrm{rank}(\pi(f)) = \dim(\mathrm{image}(\pi(f)))$.

In this paper, we replace the condition $\mu(B_f) < \infty$ by the condition (0.2.1). More precisely, we define for f in $L^2(G) \cap L^1(G)$ the number $\hat{\nu}(f)$ by

$$\hat{\nu}(f) = \int_{\widehat{G}} \mathrm{rank}(\pi(f))\, d\hat{m}(\pi), \tag{0.2.2}$$

and we say that a locally compact unimodular type I group G satisfies (Q.U.P.) if for any function $f \in L^2(G)$,

$$m(A_f) < m(G) \text{ and } \hat{\nu}(f) < \infty \Rightarrow f = 0 \quad m\text{-a.e.} \tag{0.2.3}$$

There is a third less restrictive candidate in the literature for the second inequality in the definition of Q.U.P., namely

$$\hat{m}(B_f) < \infty.$$

But this last definition leads to very difficult still unsolved problems (see §1.6).

0.3. In the first section, we prove that the property $\hat{\nu}(f) < \infty$ is equivalent to the existence of an $h \in L^2(G)$ such that $h = h^*$ and $h * f = f$ (Proposition 1.1). In fact, for a unimodular group G "many" functions f in $L^2(G)$ with the property $\hat{\nu}(f) < \infty$ do exist (Theorem 1.4). We show then that any unimodular type I group G, which contains a closed noncompact connected subset satisfies (Q.U.P.) (see Theorem 1.3). The proof of this theorem is of course an adaptation of the proofs in [Amr,Ber] (as in [Hog]).

We conclude section 1 with another type of weak Q.U.P. property for some particular group (Theorem 1.8).

0.4. We say that a locally compact unimodular type I group G has the weak Paley-Wiener property, if any distribution T on G with compact support, whose Fourier transform vanishes on a subset of positive Plancherel measure, must be 0.

Recently, Ronald L. Lipsman and Jonathan Rosenberg have shown that nilpotent Lie groups have the weak Paley-Wiener property (see [Lip,Ros]). We shall present in our second section a different proof of this result, based on the notion of adapted Fourier transform (§2.2).

1. Q.U.P. for unimodular groups

a. A general Q.U.P.

1.1. Proposition. *Let G be a locally compact unimodular group of type* I. *Let f be in $L^2(G) \cap L^1(G)$. Then the number $\hat{\nu}(f) = \int_{\widehat{G}} \operatorname{rank}(\pi(f))\, d\hat{m}(\pi)$ is finite if and only if there exists $h = h^*$ in $L^2(G)$, such that $h * f = f$.*

Proof. Suppose that $\hat{\nu}(f)$ is finite. Then $\operatorname{rank}(\pi(f))$ must be finite for almost all π. Hence $\operatorname{image}(\pi(f * f^*)) = \operatorname{image}(\pi(f))$, π-a.e. in $\widehat{G}$. Let $h_0 = f * f^*$ and for k in $\mathbb{N}$, let $h_k = (h_{k-1})^{1/2}$, where we consider h_k as an element of the C^*-algebra $C^*(G)$ of G. The limit $h = \lim_{k\to\infty} h_k$ exists in the von Neumann algebra $Vn(G)$ of G and its Fourier transform $\hat{h}$ is given for almost all π in $\widehat{G}$ by

$$\hat{h}(\pi) = \lim_{k\to\infty} \hat{h}_k(\pi) = \lim_{k\to\infty} (\pi(h_{k-1}))^{1/2} = H(\pi),$$

where $H(\pi)$ is the orthogonal projection onto the image of $\pi(f)$.

The field of operators $H : \pi \mapsto H(\pi) \in \operatorname{End}(\mathcal{H}_\pi)$ is measurable and is of course contained in the von Neumann algebra of the decomposable operators on $L^2(\widehat{G})$ (see [Dix(1)], 18.8). Furthermore, since

$$\int_{\widehat{G}} \operatorname{tr}(H(\pi)^2)\, d\hat{m}(\pi) = \int_{\widehat{G}} \operatorname{rank}(\pi(f))\, d\hat{m}(\pi) < \infty,$$

we see that H belongs to $L^2(\widehat{G})$ and so h to $L^2(G)$. Since $H(\pi) \circ \pi(f) = \pi(f)$ for almost all $\pi \in \widehat{G}$, we have that $h * f = f$.

Conversely, suppose that $h * f = f$, for some $h = h^*$ in $L^2(G)$. Then since $\hat{h}(\pi) \circ \hat{f}(\pi) = \pi(f)$ for almost all π in $\widehat{G}$, it follows that $\operatorname{image}(\pi(f))$ is contained in the eigenspace of $\hat{h}(\pi)$ for the eigenvalue 1. Hence $\operatorname{rank}(\pi(f)) \leq \operatorname{tr}(\hat{h}(\pi)^2)$ and

$$\hat{\nu}(f) = \int_{\widehat{G}} \operatorname{rank}(\pi(f))\, d\hat{m}(\pi) \leq \int_{\widehat{G}} \operatorname{tr}(\hat{h}(\pi)^2)\, d\hat{m}(\pi) = \|h\|_2^2 < \infty. \qquad \square$$

The following lemma can also be found in a slightly different form in [Hog].

1.2. Lemma. *Let G be a unimodular locally compact group which contains a closed noncompact connected subset. Let $V \subset W$ be two measurable subsets of G, such that $0 \neq m(V)$, $m(W) < \infty$. Then for any $\varepsilon > 0$, there exists $x \in G$, such that*

$$m(V) - 2\varepsilon < m(V \cdot x \cap W) < m(V) - \varepsilon.$$

Proof. Clearly the connected component of the identity G_0 is not compact. Denoting the characteristic function of a subset A of G by 1_A, we have for x in G,

$$m(V \cdot x \cap W) = \int_G 1_{V \cdot x}(y) 1_W(y)\, dm(y) = (1_V)^* * 1_W(x).$$

The functions $(1_V)^*$ and 1_W are in $L^2(G)$ and so $\phi = (1_V)^* * 1_W$ is a continuous function which vanishes at infinity. This tells us, that $\lim_{G_0 \ni x \to \infty} \phi(x) = 0$. Since G_0 is connected, the image of the restriction of ϕ to G_0 contains the connected subset $]0, m(V)[$ of $\mathcal{R}$. □

1.3. Theorem. *Let G be a locally compact unimodular group containing a closed noncompact connected subset. Let f be in $L^2(G)$ such that $m(A_f) < m(G)$ and such that there exists h in $L^2(G)$ with $h * f = f$. Then $f = 0$ m-a.e.*

Proof. Suppose that $f \neq 0$. We observe that for any $x \in G$, $h * f_x = f_x$.

Now let $C = \mathrm{supp} f = A_f$ and let $x_0 = e$, the identity element of G. Using Lemma 1.2 we can easily find a sequence $x_0, x_1, \ldots,$ in G, such that

$$m(C) - \frac{1}{2^{k-1}} < m(C \cdot x_k \cap C_{k-1}) < m(C) - \frac{1}{2^k}, \qquad k \in \mathbb{N}, \tag{1.3.1}$$

where for $k \geq 1$, $C_{k-1} = \bigcup_{i=0}^{k-1} C \cdot x_i \supset C$.

Let $C_\infty = \bigcup_{k=0}^{\infty} C_k$. We have that

$$\begin{aligned} m(C_{k+1}) &= m(C_k \cup C \cdot x_{k+1}) = m(C_k) + m(C \cdot x_{k+1} \backslash C_k) \\ &= m(C_k) + m(C \cdot x_{k+1}) - m(C \cdot x_{k+1} \cap C_k) \leq m(C_k) + \frac{1}{2^k} \end{aligned}$$

by (1.3.1) and the identity $m(C \cdot x) = m(C)$ for all x in G. Hence, $m(C_{k+1}) \leq m(C) + \sum_{i=0}^{k} \frac{1}{2^i}$, for all k in $\mathbb{N}$ and so $m(C_\infty) < \infty$.

Let $\phi = 1_{C_\infty}$ and consider the operator $K : L^2(G) \to L^2(G)$, $g \mapsto \phi \cdot h * g$. Since

$$K(g)(x) = \phi(x) \int_G h(y) g(y^{-1}x)\, dm(y) = \int_G \phi(x) h(xy^{-1}) g(y)\, dm(y),$$

K is a kernel operator with kernel function: $(x, y) \mapsto \phi(x) h(xy^{-1})$. But this kernel function is in $L^2(G \times G)$ and thus K is a Hilbert-Schmidt operator and hence is compact.

We have that $\mathrm{supp} f_{x_k} = C \cdot x_k \subset C_\infty$. Whence the functions f_{x_k} are all eigenvectors of K for the eigenvalue 1. On the other hand, again by (1.3.1),

$$m(\mathrm{supp} f_{x_k} \backslash \bigcup_{i=0}^{k-1} \mathrm{supp} f_{x_i}) \neq 0.$$

The functions f_{x_j} are thus all linearly independent in $L^2(G)$ and so the compact operator K has an eigenspace of infinite dimension, which is impossible. We see that f must be the zero function in $L^2(G)$. □

1.4. If G is unimodular, it is easy to construct by the method of [Dix(2)] elements h, f in $L^2(G)$ such that $h \neq 0$ and $h * f = f$.

1.4. Theorem. *Let G be a locally compact unimodular group. The functions f in $L^2(G)$ such that there exists $h = h^*$ in $L^2(G)$ so that $h * f = f$ are dense in $L^2(G)$.*

Proof. Indeed, let $g = g^*$ in $L^1(G) \cap L^2(G)$. Define for $\lambda \in \mathbb{R}$, $e(i\lambda g)$ by

$$e(i\lambda g) = i\lambda g + \frac{1}{2}(i\lambda g)^{2*} + \frac{1}{3!}(i\lambda g)^{3*} + \cdots = \sum_{j=1}^{\infty} \frac{1}{j!}(i\lambda L_g)^{j-1}(i\lambda g)$$

where L_g denotes the selfadjoint operator $L_g(k) = g * k$ on $L^2(G)$. Let $q(\lambda)$ be the function $\frac{e^{i\lambda}-1}{i\lambda}$, for λ in $\mathbb{R}$. Then the operator $\sum_{j=1}^{\infty} \frac{1}{j!}(i\lambda L_g)^{j-1}$ is $q\{L_g\}$ in the sense of functional calculus and its norm $\| \sum_{j=1}^{\infty} \frac{1}{j!}(i\lambda L_g)^{j-1} \|_{\mathrm{op}}$ is less than $\|q\|_\infty$. Whence:

$$e(i\lambda g) \in L^2(G) \quad \text{and} \quad \|e(i\lambda g)\|_2 \leq \|g\|_2 \|q\|_\infty (1 + |\lambda|), \quad \forall \lambda \in \mathbb{R}.$$

For any function ϕ in $C^3(\mathbb{R})$, with compact support, such that $\phi(0) = 0$, we have $|\hat{\phi}(\lambda)| \|e(i\lambda g)\|_2 \leq C(1 + |\lambda|)^{-2}$ for some contant C independent of λ and so the integral

$$\phi\{g\} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\phi}(\lambda) e(i\lambda g)\, d\lambda$$

converges in $L^2(G)$. Furthermore we have

$$\phi\{g\} * k = \phi\{L_g\}(k), \qquad k \in L^2(G). \tag{1.4.1}$$

Indeed, $e(i\lambda g) * k = (\exp(L_{i\lambda g}) - 1)(k)$ and $\| \exp(L_{i\lambda g}) \|_{\mathrm{op}} = 1$, for all λ in $\mathbb{R}$, hence for any character μ of the (abelian) C^*-algebra generated by L_g, we have

$$\mu\left(\int_{\mathbb{R}} \hat{\phi}(\lambda) e(L_{i\lambda g})\, d\lambda \right) = \int_{\mathbb{R}} \hat{\phi}(\lambda) \exp(i\lambda \mu(L_g))\, d\lambda = 2\pi \phi(\mu(L_g)),$$

by Fourier's inversion formula.

Choose now C^3 functions $\varphi, \phi : \mathbb{R} \to \mathbb{R}$ such that φ and ϕ vanish on a neighbourhood of 0 and such that $\phi\varphi = \varphi$. Let

$$h = \int_{\mathbb{R}} \hat{\phi}(\lambda) \exp(i\lambda g)\, d\lambda, \qquad f = \int_{\mathbb{R}} \hat{\varphi}(\lambda) \exp(i\lambda g)\, d\lambda \in L^2(G).$$

Then $h * f * k = \phi\{L_g\} \circ \varphi\{L_g\}(k) = \phi\varphi\{L_g\}(k) = \varphi\{L_g\}(k) = f * k$, for all k in $L^2(G)$. Whence $h * f = f$. In order to prove density, let k be any element in $L^2(G)$ and let $\varepsilon > 0$. Choose $g = g^*$ in $L^1(G)$ continuous with compact support, such that $\|g\|_1 = 1$ and $\|g^{3*} * k - k\|_2 \leq \varepsilon$. Take a function φ_0 in $C^3(\mathbb{R})$ with $\varphi_0(t) = t^3$ in a neighbourhood of $]-1, 1[$. Then $\varphi_0\{g\} = g^{*3}$. Choose φ in $C^3(\mathbb{R})$ so that $\|\varphi - \varphi_0\|_\infty < \varepsilon$ and so that φ vanishes in a neighbourhood of 0. Then

$$\begin{aligned} \|\varphi\{g\} * k - g^{3*} * k\|_2 &= \|\varphi\{g\} * k - \varphi_0\{g\} * k\|_2 = \|(\varphi\{L_g\} - \varphi_0\{L_g\})(k)\|_2 \\ &\leq \|\varphi\{L_g\} - \varphi_0\{L_g\}\|_{\mathrm{op}} \|k\|_2 \leq \|\varphi - \varphi_0\|_\infty \|k\|_2 < \varepsilon \|k\|_2. \end{aligned}$$

Let $f = \varphi\{g\}$, f is in $L^2(G)$. We then have

$$\|f * k - k\|_2 \leq (1 + \|k\|_2)\varepsilon, \qquad h * f * k = f * k, \quad \text{for } h = \phi\{g\} \in L^2(G). \qquad \square$$

1.5. Corollary. *Let G be a unimodular group containing a closed noncompact connected subset. Let $f = f^*$ be in $L^2(G) \cap L^1(G)$ and let φ be in $C^3(\mathbb{R})$ which is zero in a neighbourhood of 0. If $m(A_{\varphi\{f\}})$ is finite, then $\varphi\{f\} = 0$.*

The proof is an immediate consequence of Theorem 1.3 and the proof of Theorem 1.4.

b. A weak Q.U.P. for special groups.

1.6. Let G be a connected simply connected nilpotent Lie group. The problem whether for such groups the condition $f \in L^2(G)$, $m(A_f)\hat{m}(B_f) < \infty$ implies $f = 0$ is still not settled, not even in low dimensions except for some special Heisenberg type groups (see [Ech,Kan,Kum]). Nevertheless, we shall close this section with a weaker result, for a relatively larger class of groups.

From now on, G is a locally compact group containing a nontrivial central vector group Z. Every π in $\widehat{G}$ restricts on Z to a character χ_π.

1.7. Definition. We say that a subset $\mathcal{B}$ of $\widehat{G}$ is *bounded in* $\widehat{Z}$, if there exists a bounded subset $\mathcal{B}_0$ in $\widehat{Z}$ such that for every π in $\mathcal{B}$, χ_π is in $\mathcal{B}_0$.

We shall prove here the vanishing of any function f in $L^2(G)$, such that $m(A_f) < \infty$ and B_f bounded in $\widehat{Z}$. This requirement on f is clearly stronger than the condition $m(A_f)\hat{m}(B_f) < \infty$, thus we say that Theorem 1.8 is a weak Q.U.P. for G.

1.8. Theorem. *Let G be a unimodular amenable separable group containing a nontrivial central vector group Z. Let f be in $L^2(G)$ such that $m(A_f) < \infty$ and such that B_f is bounded in $\widehat{Z}$. Then $f = 0$.*

Proof. Since, for any g in $L^1(G)$,

$$\int_G g(x)\,dm(x) = \int_{G/Z} (\int_Z g(xz)\,dm_Z(z))\,dm_{G/Z}(x),$$

it follows from Fubini's theorem, that for almost every x in G, the function $z \mapsto {}_xf_{|Z}(z) := f(xz)$ is in $L^2(Z)$ and $A({}_zf_{|Z})$ is of bounded Lebesgue measure. The fact that B_f is bounded in $\widehat{Z}$ implies that there exists a bounded subset $\mathcal{B}_0$ in $\widehat{Z}$, such that $B_f \subset \{\pi \in \widehat{G}; \chi_\pi \in \mathcal{B}_0\}$. Let

$$\widehat{G}_{\mathcal{B}_0} := \{\pi \in \widehat{G},\ \text{such that } \chi_\pi \notin \mathcal{B}_0\}.$$

Then $\widehat{G}_{\mathcal{B}_0}$ has empty intersection with B_f. Fix now χ in $\widehat{Z}\backslash\mathcal{B}_0$ and let $\rho = \operatorname{ind}_Z^G \chi$. Since G is amenable, the kernel of ρ in $C^*(G)$ is the intersection of all the $\ker_{C^*(G)} \pi$ for which $\chi_\pi = \chi$. It follows that f is in $\ker_{C^*(G)} \rho$, i.e. $\rho(f) = 0$. The kernel of ρ in $L^1(G)$ is also given by

$$\begin{aligned}\ker_{L^1(G)} \rho =& \{g \in L^1(G); \text{for almost every } x \in G,\\ & \qquad {}_xg_{|Z} \in L^1(Z) \text{ and } ({}_xg_{|Z})^\wedge(\chi) = 0\}\end{aligned}$$

(see [Lud]). Hence for almost all x in G, $({}_xf_{|Z})^\wedge(\chi) = 0$. If we choose a countable dense subset $\mathcal{S}$ in $\widehat{Z}\backslash\mathcal{B}_0$, then we see that there exists a null set $\mathcal{N}$ in G, such that

(i) for every $x \notin \mathcal{N}$, the function ${}_xf_{|Z}$ is in $L^2(Z)$,
(ii) the support of ${}_xf_{|Z}$ is of bounded Haar measure and
(iii) $({}_xf_{|Z})^\wedge(\chi) = 0$ for all χ in $\mathcal{S}$, hence for all χ in $\widehat{Z}\backslash\mathcal{B}_0$.

Thus for any $x \notin \mathcal{N}$, $({}_xf_{|Z})^\wedge$ has its support contained in the bounded subset $\mathcal{B}_0$, it is in $L^2(\widehat{Z})$ and its Fourier transform ${}_xf_{|Z}$ vanishes on a subset of positive Lebesgue measure, since $m(A_{{}_xf}) < \infty$. Whence Q.U.P. for the abelian group Z tells us that ${}_xf_{|Z} = 0$, for all $x \notin \mathcal{N}$. Finally $f = 0$. ∎

2. A weak Paley-Wiener property for nilpotent Lie groups

2.0. Let G be a nilpotent Lie group and let T be a distribution on G with compact support. For any f in the Schwartz space $S(G)$ of the rapidly decreasing C^∞ functions on G, the function $T*f$, defined by $T*f(x) = \langle T, l_x(\tilde{f})\rangle$ $(x \in G)$, where $\tilde{f}(x) = f(x^{-1})$, is also in $S(G)$ and $f \mapsto T*f$ defines a continuous endomorphism of $S(G)$. We can define for π in $\widehat{G}$ the operator $\pi(T)$ which acts on the space $\mathcal{H}_\pi^\infty$ of the C^∞-vectors of $\mathcal{H}_\pi$ in the following way.

By a result of Howe (see [Cor,Gre] 4.2.1), the convolution algebra $S(G)$ acts algebraically irreducibly on $\mathcal{H}_\pi^\infty$ and so it is easy to see that

$$\pi(T)(\pi(f)\xi) := \pi(T*f)\xi, \qquad f \in S(G), \xi \in \mathcal{H}_\pi,$$

is well defined and that $\pi(T)$ is even a continuous linear endomorphism of $\mathcal{H}_\pi^\infty$. We can thus define the Fourier transform $\widehat{T}$ of T by setting $\widehat{T}(\pi) = \pi(T)$ in $\mathrm{End}(\mathcal{H}_\pi^\infty)$, π in $\widehat{G}$.

Lipsman and Rosenberg ([Lip,Ros]) have recently proved a conjecture of Moss (see [Mos]), which said that if $\widehat{T}$ vanishes on a subset of positive Plancherel measure, then $T = 0$. Recently a new proof of this result has appeared in a preprint of G. Garimella (see [Gar]). We shall present here another proof of this conjecture which is based on the notion of the adapted Fourier transform (see [Arn,Cor], [Arn,Gut] and [Lud,Zah]). Let us recall that Kirillov's theory allows us to identify $\widehat{G}$ with the space $\mathfrak{g}^*/G$ of the coadjoint orbits of G on the dual vector space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G. Denote by K the Kirillov mapping

$$K : \mathfrak{g}/G \to \widehat{G}, \qquad l \mapsto \pi_l = K(\mathrm{Ad}^*(G)l).$$

The relation between $\Omega = \mathrm{Ad}^*(G)l$ and $\pi = K(\Omega)$ is given by Kirillov's character formula

$$\mathrm{tr}(\pi(f)) = \int_\Omega (f \circ \exp)^\wedge(\xi)\, d\mu(\xi) \tag{2.0.1}$$

for any f in $S(G)$, where $d\mu$ denotes the G-invariant Kirillov measure on Ω.

2.1. Pukanszky has given the following description of the orbit space $\mathfrak{g}^*/G$. Let $\mathfrak{B} = \{q_1, \ldots, q_n\}$ be a Jordan-Hölder basis of the $\mathfrak{g}$-module $\mathfrak{g}^*$. There exists a family of r polynomial functions p_k, $1 \le k \le r$, on $\mathfrak{g}^*$ and for every j in $\{1, \ldots, r\}$ an index set $e_j = \{i_1(j), \ldots, i_{d(j)}(j)\} \subset \{1, \ldots, n\}$ with the following property:

Let $\mathfrak{g}_j^* = \{l \in \mathfrak{g}^*; p_1(l) = \ldots = p_{j-1}(l) = 0,\ p_j(l) \neq 0\}$ $(j = 1, \ldots, r)$ (here $p_0 = 0$). The substs $\mathfrak{g}_j^*$ form a G-invariant partition of $\mathfrak{g}^*$. There exists for every j a family of functions $p_1^j, \ldots, p_n^j$ on $\mathfrak{g}_j^* \times \mathbb{R}^{d(j)}$ which are polynomials in $Z \in \mathbb{R}^{d(j)}$ with rational coefficients in $l \in \mathfrak{g}_j^*$ and the denominators of these rational functions in l are powers of p_j. These p_i^j, $i = 1, 2, \ldots$, have the following properties:

(i) $p_k^j(l, (z_1, \ldots, z_{d(j)}))$ does only depend on l and on $z_1, \ldots, z_{k'}$, where $i_{k'}(j) \le k < i_{k'+1}(j)$.
(ii) $p_{i_k}^j = z_k, k = 1, \ldots, d(j), i_k \in e_j$.
(iii) $\mathrm{Ad}^*(G)l = \{P^j(l, z) := \sum_{k=0}^n p_k^j(l, z)q_k, z \in \mathbb{R}^{d(j)}\}$, $l \in \mathfrak{g}_j^*$.

Now let $d = d(1)$, $e = e_1 = \{i_1, \ldots, i_d\}$ and let $\mathfrak{V} = \sum_{i \notin e} \mathbb{R}q_i$ and $\mathfrak{V}_1 = \mathfrak{V} \cap \mathfrak{g}_1^*$. Then for any $l \in \mathfrak{V}_1$ we have $l = P^1(l, 0)$ and the mapping

$$\mathfrak{V}_1 \to \mathfrak{g}_1^*/G, \qquad l \mapsto \mathrm{Ad}^*(G)P^1(l, 0)$$

is a bijection. In particular $\mathfrak{V}_1$ is Zariski-open in $\mathfrak{V}$, $\widehat{G}_1 := K(\mathfrak{g}_1^*/G) \subset \widehat{G}$ carries the Plancherel measure and there exists a polynomial function Q on $\mathfrak{V}$ such that the measure $|Q|dl$ corresponds to the Plancherel measure on $\widehat{G}$ under the bijection $\mathfrak{V}_1 \to \widehat{G}_1 : l \mapsto \pi_l$. In fact for any measurable function ϕ on $\mathfrak{g}^*$, we have

$$\int_{\mathfrak{g}^*} \phi(l)\, dl = \int_{\mathfrak{V}_1} \left(\int_{\mathbb{R}^{d(1)}} \phi(P^1(l,Z))\, dZ \right) |Q(l)|\, dl. \tag{2.1.1}$$

2.2. Let us now define the adapted Fourier transform (see [Lud,Zah]).

There exists a G-invariant Zariski-open subset $\mathfrak{g}_0^*$ of $\mathfrak{g}_1^*$ and a continuous function

$$a : G \times \mathfrak{g}_0^* \to \mathbb{R}, \qquad (x,l) \mapsto a(x,l) = \sum_\alpha a_\alpha(l) x^\alpha$$

which is polynomial in $x \in G$ with rational coefficients $a_\alpha = \frac{Q_\alpha}{Q_0}$, whose common denominator Q_0 does not vanish on $\mathfrak{g}_0^*$ and which replaces the usual bracket $\langle l, x\rangle$ on $\mathfrak{g}^* \times \mathfrak{g}$, i.e., if we define the adapted Fourier transform $\hat{f}^a$ of an $L^1(G)$-function f by

$$\hat{f}^a(l) = \int_G f(x) e^{-ia(x,l)} dm(x), \qquad l \in \mathfrak{g}_0^*,$$

then $\hat{f}^a$ is a continuous bounded function on $\mathfrak{g}_0^*$ and, for any $\pi = K(\mathrm{Ad}^*(G)l)$ in $\widehat{G}_0 = K(\mathfrak{g}_0^*/G)$, for any C^∞-function f with compact support, we have

$$\mathrm{tr}(\pi(f) \circ \pi(f)^*) = \int_{\mathrm{Ad}^*(G)l} |\hat{f}^a(\xi)|^2\, d\mu(\xi), \tag{2.2.1}$$

where $d\mu$ is as in (2.0.1). Formula (2.2.1) allows us to deduce from $\hat{f}^a = 0$ on $\mathrm{Ad}^*(G)l$ that $\pi(f) = 0$, whereas $\hat{f}(\mathrm{Ad}^*(G)l) = 0$ generally does not imply the vanishing of $\pi(f)$ (see [Lud]).

2.3. We shall give now a new proof of theorem 2.16 in [Lip,Ros]. We need first a technical result on analytic functions, for which we could not find a reference.

2.3 Lemma. *Let $\mathcal{U}$ be an open connected subset of $\mathbb{R}^n$, let $f : \mathcal{U} \to \mathbb{C}$ be an analytic function. If f vanishes on a subset of positive Lebesgue measure, then $f = 0$.*

Proof. (Induction on n) The lemma is of course trivial for $n = 1$. Suppose now $n > 1$. Let $N = \{x \in \mathcal{U}, f(x) = 0\}$. Choose a box $V = [a,b] \times \overline{V}$ in $\mathcal{U}$ such that $\int_V 1_N(y)\, dy \neq 0$. Fubini's theorem implies that $\int_{\overline{V}} 1_N(y_1, \overline{y})\, d\overline{y} \neq 0$ for all y_1 in a subset I of positive Lebesgue measure in $[a,b]$. For such a y_1, the analytic function $\overline{y} \mapsto f(y_1, \overline{y})$ vanishes on a set of positive Lebesgue measure and so must be 0 by the induction hypothesis. Hence $f(y_1, \overline{y}) = 0$ for any y_1 in I, for every $\overline{y}$ in $\overline{V}$. Since I is of positive Lebesgue measure, we see that f is identically zero on V. □

2.4 Theorem. *Let G be a connected simply connected nilpotent Lie group. Let T be a distribution on G with compact support such that $\widehat{T}$ vanishes in $\widehat{G}$ on a subset of positive Plancherel measure. Then $T = 0$.*

Proof. Let $\mathfrak{W}'$ be a measurable subset of positive Plancherel measure in $\widehat{G}$ on which $\widehat{T}$ vanishes. Let $\mathfrak{W} = \{l \in \mathfrak{V}_1; \pi_l \in \mathfrak{W}'\}$. Since the Plancherel measure is concentrated on $K(\mathfrak{g}_0^*/G)$, and since it corresponds to $|Q|$ times Lebesgue measure on $\mathfrak{V}_1$, we see that $\mathfrak{W}$ is of positive Lebesgue measure and so $\mathcal{W} = \mathrm{Ad}^*(G)\mathfrak{W}$ is of positive Lebesgue measure in $\mathfrak{g}^*$ by (2.1.1).

Let g be any C^∞ function with compact support on G and let $f = T * g$. The function f is itself a C^∞ function whose support is contained in $\operatorname{supp} T \cdot \operatorname{supp} g$, whence is compact. Since $\pi(f) = \pi(T * g) = \pi(T) \circ \pi(g)$, for any π in $\widehat{G}$, we see that $\hat{f}$ vanishes on a set which contains $\mathfrak{W}'$. If we show that $f = 0$, then $\langle T, \tilde{g} \rangle = T * g(e) = 0$, and so T must be 0.

In order to prove that $f = 0$, we consider $\mathfrak{g}_0^* \subset \mathfrak{g}_{0\mathbb{C}}^* = \{\xi \in \mathfrak{g}_{\mathbb{C}}^*; Q_0(\xi) \neq 0\}$. It is easy to see that $\mathfrak{g}_{0\mathbb{C}}^*$ is pathwise connected and so is connected. Since the support of f is compact, the integral

$$\hat{f}^a(\xi) = \int_G f(x) e^{-ia(x,\xi)}\, dx$$

exists for any ξ in $\mathfrak{g}_{0\mathbb{C}}^*$ and the function $\hat{f}^a$ is even holomorphic on $\mathfrak{g}_{0\mathbb{C}}^*$, indeed, for any ξ, η in $\mathfrak{g}_{0\mathbb{C}}^*$,

$$\frac{d}{dz} \hat{f}^a(\xi + z\eta)_{|z=0} = \int_G \left(-i \frac{d}{dz} a(x, \xi + z\eta)_{|z=0} \right) f(x) e^{-ia(x,\xi)}\, dx$$

does exist and is continuous in ξ. Since $\hat{f}$ vanishes on $\mathfrak{W}'$, we have $\pi_l(f) = 0$ for any l in $\mathfrak{W}$ and so by (2.2.1) $\hat{f}^a$ is 0 on $\mathcal{W}$, which is of positive Lebesgue measure. Hence, by Lemma 2.3, the holomorphic function $\hat{f}^a$ is identically 0 in $\mathfrak{g}_{0\mathbb{C}}^*$, since $\mathfrak{g}_{0\mathbb{C}}^*$ is connected.

Thus $\hat{f}$ vanishes on $K(\mathfrak{g}_0^*/G)$ which is dense in $\widehat{G}$. The injectivity of the Fourier transform implies that $f = 0$. □

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