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HYPERSURFACES IN A SPHERE WITH CONSTANT MEAN CURVATURE

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Abstract. Let M^n be a closed hypersurface of constant mean curvature immersed in the unit sphere S^{n+1}. Denote by S the square of the length of its second fundamental form. If $S < 2\sqrt{n-1}$, M is a small hypersphere in S^{n+1}. We also characterize all M^n with $S = 2\sqrt{n-1}$.

1. Introduction

Let M^n be a closed submanifold with parallel mean curvature vector field immersed in the unit sphere S^{n+p}. Denote by H the length of the mean curvature vector field and by S the square of the length of the second fundamental form of M^n. It is important to characterize those M immersed as n-spheres in S^{n+p} by H and S.

When M is minimal, J. Simons [9] obtained a pinching constant $n/(2-1/p)$ of S and Chern-do Carmo-Kobayashi [3] showed that it is sharp and characterized all M with $S = n/(2-1/p)$. M. Okumura [6, 7] first discussed the general case and gave a pinching constant of S, but it is not sharp. Recently the sharp ones were obtained by H. Alencar-M. do Carmo [1] for $p = 1$, W. Santos [8] for $p > 1$ and H. W. Xu [11] for $p \geq 1$ respectively. But all of them were expressed by the mean curvature H. S. T. Yau [12] obtained a pinching constant for $p > 1$ which depended only on n and p. H. W. Xu [10] improved Yau's result, but far from sharpness.

In the present paper, we shall give a pinching constant for $p = 1$ which depends only on n and show the sharpness of it. More precisely, we want to prove the following theorems:

Theorem A. *Let M^n be a hypersurface of constant mean curvature immersed in S^{n+1} with constant length of the second fundamental form. Then:*

(1) *If $S < 2\sqrt{n-1}$, M^n is locally a piece of small hypersphere $S^n(r)$ of radius $r = \sqrt{n/(n+S)}$.*
(2) *If $S = 2\sqrt{n-1}$, M is locally a piece of either $S^n(r_0)$ or $S^1(r)\times S^{n-1}(s)$ where $r_0^2 = n/(n+2\sqrt{n-1}), r^2 = 1/(\sqrt{n-1}+1)$ and $s^2 = \sqrt{n-1}/(\sqrt{n-1}+1)$.*

Theorem A′. *Let M^n be a closed hypersurface of constant mean curvature immersed in S^{n+1}. Then:*

(1) *If $S < 2\sqrt{n-1}$, M^n is a small hypersphere $S^n(r)$ of radius $r = \sqrt{n/(n+S)}$.*

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(2) *If* $S = 2\sqrt{n-1}$, *M is either a small hypersphere* $S^n(r_0)$ *or a* $H(r)$*-torus* $S^1(r) \times S^{n-1}(s)$, *where* r_0, r *and* s *are taken as before.*

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2. Proof of the theorems

Let M be a closed hypersurface immersed in the unit sphere S^{n+1}. Take a local orthonormal coframe field $\{\omega_i\}_{i=1}^n$ on M. Then the second fundamental form can be expressed as $L = (h_{ij})_{n\times n}$. The mean curvature H and the square of the length of the second fundamental form S are defined by $H = \frac{1}{n}\sum_{(i)} h_{ii}$, $S = \sum_{(i,j)}(h_{ij})^2$.

From now on, we shall always use $i, j, k, \dots$ for indices running from 1 to n.

Denote the covariant differentials of $\{h_{ij}\}$ by $\{h_{ijk}\}$ and $\{h_{ijkl}\}$. Then the Laplacian of h_{ij} is defined by $\Delta h_{ij} = \sum_{(k)} h_{ijkk}$. It follows that

$$\sum_{(i,j)} h_{ij}\Delta h_{ij} = nS + nHf - n^2H^2 - S^2, \tag{1}$$

where $f = \operatorname{Tr} L^3$ (cf. e.g. [2] or [7]).

M. Okumura [7] established the following lemma (see also [1] or [11]).

Lemma. *Let* $\{a_i\}_{i=1}^n$ *be a set of real numbers satisfying* $\sum_{(i)} a_i = 0$, $\sum_{(i)} a_i^2 = t^2$, *where* $t \geq 0$. *Then we have*

$$-\frac{n-2}{\sqrt{n(n-1)}}t^3 \leq \sum_{(i)} a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}t^3, \tag{2}$$

and equalities hold if and only if at least $(n-1)$ *of the* a_i*'s are equal to one another.*

Suppose that $\lambda_1, \lambda_2, \dots, \lambda_n$ are the principal curvatures of M. Then we have

$$nH = \sum_{(i)} \lambda_i, \quad S = \sum_{(i)} \lambda_i^2, \quad f = \sum_{(i)} \lambda_i^3. \tag{3}$$

Set $\widetilde{S} = S - nH^2$, $\tilde{f} = f - 3HS + 2nH^3$ and $\tilde{\lambda}_i = \lambda_i - H$ $(1 \leq i \leq n)$. Then (3) changes into

$$0 = \sum_{(i)} \tilde{\lambda}_i, \quad \widetilde{S} = \sum_{(i)} \tilde{\lambda}_i^2, \quad \tilde{f} = \sum_{(i)} \tilde{\lambda}_i^3. \tag{4}$$

By applying Okumura's Lemma to $\tilde{f}$ in (4), we have

$$\tilde{f} \geq -\frac{n-2}{\sqrt{n(n-1)}}\widetilde{S}\sqrt{\widetilde{S}} \iff f \geq 3HS - 2nH^3 - \frac{n-2}{\sqrt{n(n-1)}}\widetilde{S}\sqrt{\widetilde{S}}.$$

Substituting this into (1), we have

$$\sum_{(i,j)} h_{ij}\Delta h_{ij} \geq \widetilde{S}\left\{n - (\widetilde{S} - nH^2) - (n-2)H\sqrt{\frac{n}{n-1}\widetilde{S}}\right\}. \tag{5}$$

Consider the quadratic form $Q(u,t) = u^2 - \frac{n-2}{\sqrt{n-1}}ut - t^2$. By the orthogonal transformation

$$\begin{cases} \tilde{u} = \dfrac{1}{\sqrt{2n}}\{(1+\sqrt{n-1})u + (1-\sqrt{n-1})t\}, \\ \tilde{t} = \dfrac{1}{\sqrt{2n}}\{(\sqrt{n-1}-1)u + (\sqrt{n-1}+1)t\}, \end{cases}$$

$Q(u,t)$ turns into $Q(u,t) = \frac{n}{2\sqrt{n-1}}(\tilde{u}^2 - \tilde{t}^2)$, where $\tilde{u}^2 + \tilde{t}^2 = u^2 + t^2 = S$.

Take $t = \sqrt{\widetilde{S}}$ and $u = \sqrt{n}H$ in $Q(u,t)$, and substitute it into (5). We can see

$$\sum_{(i,j)} h_{ij}\Delta h_{ij} \geq \widetilde{S}\left(n - \frac{n}{2\sqrt{n-1}}S + \frac{n}{\sqrt{n-1}}\tilde{u}^2\right) \geq \widetilde{S}\left(n - \frac{n}{2\sqrt{n-1}}S\right). \tag{6}$$

Therefore we have

$$\frac{1}{2}\Delta S = \sum_{(i,j,k)} h_{ijk}^2 + \sum_{(i,j)} h_{ij}\Delta h_{ij} \geq \widetilde{S}\left(n - \frac{n}{2\sqrt{n-1}}S\right). \tag{7}$$

Theorem A. *Let M^n be a hypersurface of constant mean curvature immersed in S^{n+1} with constant length of the second fundamental form. Then:*

(1) *If $S < 2\sqrt{n-1}$, M is locally a piece of a small hypersphere $S^n(r)$ in S^{n+1}, where $r = \sqrt{n/(n+S)}$.*
(2) *If $S = 2\sqrt{n-1}$, M is locally a piece of either $S^n(r_0)$ or $S^1(r)\times S^{n-1}(s)$, where $r_0^2 = n/(n+2\sqrt{n-1})$, $r^2 = 1/(\sqrt{n-1}+1)$ and $s^2 = \sqrt{n-1}/(\sqrt{n-1}+1)$.*

Proof. Since S is constant, the left-hand side of (7) is zero. When $S \leq 2\sqrt{n-1}$, we have

$$\widetilde{S}\left(n - \frac{n}{2\sqrt{n-1}}S\right) = 0, \qquad h_{ijk} = 0, \quad 1 \leq i,j,k \leq n. \tag{8}$$

If $S < 2\sqrt{n-1}$, we have $\widetilde{S} = 0$, which means that M is totally umbilical and hence is locally a piece of hypersphere $S^n(r)$ where $r = \sqrt{n/(n+S)}$.

Suppose $S = 2\sqrt{n-1}$. Then all of the inequalities in (5)–(7) become equal ones. Okumura's Lemma implies that at least $(n-1)$ of λ_i's are equal to one another. When $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, M is totally umbilical and hence is locally a piece of hypersphere $S^n(r)$ where $r^2 = n/(n+2\sqrt{n-1})$. When M is not totally umbilical, there are exactly $(n-1)$ of λ_i's that are equal to one another. The same arguments as those developed by Chern-do Carmo-Kobayashi (see [3], p. 68) show that M is locally a piece of $S^1(r)\times S^{n-1}(s)$ in S^{n+1}. To determine the radii r and s, we refer to the examples of K. Nomizu and B. Smyth [5], from which we have

$$H = -\frac{1}{n}\left(\frac{s}{r}\right) + \frac{n-1}{n}\left(\frac{r}{s}\right), \qquad S = \left(\frac{s}{r}\right)^2 + (n-1)\left(\frac{r}{s}\right)^2.$$

It is easy to see that

$$\left(\frac{s}{r}\right)^2 + (n-1)\left(\frac{r}{s}\right)^2 \geq 2\sqrt{n-1}$$

and equality holds if and only if $\left(\frac{s}{r}\right)^2 = \sqrt{n-1}$. Therefore we have $r^2 = \frac{1}{\sqrt{n-1}+1}$ and $s^2 = \frac{\sqrt{n-1}}{\sqrt{n-1}+1}$. □

When M is closed, the integral of the left-hand side of (7) on M is equal to zero, and so is that of the right-hand side. After the same deduction as in the proof of Theorem A, we can obtain the following:

Theorem A′. *Suppose M is a closed hypersurface of constant mean curvature immersed in S^{n+1}. Then:*

(1) *If $S < 2\sqrt{n-1}$, M is a small hypersphere $S^n(r)$, where $r = \sqrt{n/(n+S)}$.*
(2) *If $S = 2\sqrt{n-1}$, M is either a small hypersphere $S^n(r_0)$ or $S^1(r)\times S^{n-1}(s)$, where r_0, r and s are taken as in Theorem A.*

We can show an application of Theorem A′. H. W. Xu [10] proved the following:

Proposition (Xu). *Let M^n be an n-dimensional compact submanifold with parallel mean curvature vector field in S^{n+p} and $p > 1$. If*

$$S \leq \min\left\{\frac{2n}{1+\sqrt{n}}, \frac{n}{2-(p-1)^{-1}}\right\},$$

and the Gauss mapping of M is relatively affine, then M^n is a standard hypersphere in a totally geodesic S^{n+1} of S^{n+p}.

By Theorem A′, we can remove the assumption that the Gauss mapping is relatively affine. Namely we can obtain the following

Corollary. *Let M^n be an n-dimensional compact submanifold with parallel mean curvature vector field in S^{n+p} and $p > 1$. If*

$$S \leq \min\left\{\frac{2n}{1+\sqrt{n}}, \frac{n}{2-(p-1)^{-1}}\right\},$$

then M^n is a standard hypersphere in a totally geodesic S^{n+1} of S^{n+p}.

Proof. It is easy to check that $(\sqrt{n}+1)/n > 1/\sqrt{n-1}$. Therefore we have

$$\sqrt{n-1} > \frac{n}{\sqrt{n}+1} \iff 2\sqrt{n-1} > \frac{2n}{\sqrt{n}+1} \geq S.$$

□

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