

WILSON'S FUNCTIONAL EQUATION FOR VECTOR AND MATRIX FUNCTIONS

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ABSTRACT. We determine the general solution of the functional equation

$$f(x+y) + f(x-y) = A(y)f(x) \quad (x, y \in G),$$

where G is a 2-divisible abelian group, f is a vector-valued function and A is a matrix-valued function. Using this result we solve the scalar equation

$$f(x+y) + f(x-y) = g_1(x)h_1(y) + g_2(x)h_2(y) \quad (x, y \in G),$$

which contains as special cases, among others, the d'Alembert and Wilson equations and the parallelogram law.

1. INTRODUCTION

Let G be an abelian group divisible by 2 (that is, $2G = G$) and K an algebraically closed commutative field of characteristic different from 2 and 3. Throughout this paper we denote by $\mathcal{L}, \mathcal{M}, \mathcal{N}$ the sets of all solutions $f: G \rightarrow K$ of the functional equations $f(x+y) = f(x) + f(y)$, $f(x+y) = f(x)f(y)$ and $f(x+y) + f(x-y) = 2f(x) + 2f(y)$, respectively. Also, K_2 is the set of all 2×2 matrices over K , K^2 is the set of all 2×1 matrices (column vectors) over K , \mathcal{F} is the set of all vector-valued functions with linearly independent components, I is the unit matrix of K_2 and 0 (the zero of K) is also used for the zeros of G, K^2 and K_2 . If $a \in \mathcal{L}$, then a is called additive.

Clearly, if $M \in K_2$, $f \in \mathcal{F}$ and $Mf = 0$, then $M = 0$.

In this paper we determine the general solutions $f: G \rightarrow K^2$, $A: G \rightarrow K_2$ of the functional equation

$$f(x+y) + f(x-y) = A(y)f(x) \quad (x, y \in G),$$

which can be viewed as a vector analogue of Wilson's functional equation (1.1) below (cf. also Eq. (3.4) of [4]).

The main result is given in Theorem 1. Using this result we solve in Theorem 2 the scalar equation

$$f(x+y) + f(x-y) = g_1(x)h_1(y) + g_2(x)h_2(y) \quad (x, y \in G).$$

This last equation contains as special cases the Wilson equation

$$(1.1) \quad f(x+y) + f(x-y) = f(x)g(y),$$

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introduced in [9], the equations

$$(1.2) \quad f(x+y) + f(x-y) = 2f(x) + g(x)h(y),$$

$$(1.3) \quad f(x+y) + f(x-y) = f(x)g(y) + g(x)f(y),$$

$$(1.4) \quad f(x+y) + f(x-y) = f(x)f(y) + g(x)g(y),$$

which were solved in [1], [3], and the parallelogram law

$$(1.5) \quad f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

which characterizes the diagonalization of biadditive functions ([1], [2]). More general equations were solved e.g. in [4], but, as remarked in [3], to extract the solutions of (1.1)–(1.5) from [4] is not simple.

2. THE RESULTS

Theorem 1. *If $A: G \rightarrow K_2$, $f: G \rightarrow K^2$ with $f \in \mathcal{F}$ is a solution of the functional equation*

$$(2.1) \quad f(x+y) + f(x-y) = A(y)f(x) \quad (x, y \in G),$$

then

$$(2.2) \quad \begin{cases} A(y) = C[E(y) + E(-y)]C^{-1}, \\ f(x) = C[E(x)\alpha + E(-x)\beta], \end{cases}$$

where $C \in K_2$ ($\det C \neq 0$), $\alpha, \beta \in K^2$ and $E(x)$ has one of the forms

$$(2.3) \quad \begin{aligned} E(x) &= \chi(x) \begin{vmatrix} 1 & \phi(x) \\ 0 & 1 \end{vmatrix}, \\ &\begin{vmatrix} 1 & n(x) + \phi(x) \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} \phi(x) + 1 & c\phi(x)^3 + 3c\phi(x)^2 + \phi_1(x) \\ 0 & \phi(x) + 1 \end{vmatrix}, \\ &\begin{vmatrix} \chi_1(x) - [\chi_1(0) - 1][\phi_1(x) + 1] & 0 \\ 0 & \chi_2(x) - [\chi_2(0) - 1][\phi_2(x) + 1] \end{vmatrix}, \end{aligned}$$

with $c \in K$, $\phi, \phi_1, \phi_2 \in \mathcal{L}$, $\chi, \chi_1, \chi_2 \in \mathcal{M}$, $\chi(x) \not\equiv 0$, and $n \in \mathcal{N}$.

Conversely, any A, f as described above satisfy (2.1).

Note. If $f \notin \mathcal{F}$, then (2.1) reduces to a system of two Wilson equations and the solutions are obtained by application of Theorem (2.2) of [5].

Proof. Setting $y = u + v$ and $y = u - v$ in (2.1), adding the resulting equations and using (2.1) again we obtain

$$A(v)A(u)f(x) = [A(u+v) + A(u-v)]f(x).$$

Since $f \in \mathcal{F}$ this leads to the matrix d'Alembert equation

$$(2.4) \quad A(v)A(u) = A(u+v) + A(u-v).$$

Replacing y by $-y$ in (2.1) we see that $A(y)f(x) = A(-y)f(x)$ and so $A(y) = A(-y)$. Hence, interchanging u, v in (2.4), we have $A(u)A(v) = A(v)A(u)$. Since K is algebraically closed there exists an invertible matrix $C \in K_2$ such that the matrix function $B(x) := C^{-1}A(x)C$ has the form

$$B(x) = \begin{vmatrix} b_1(x) & b_0(x) \\ 0 & b_2(x) \end{vmatrix}$$

with $b_1(x) \equiv b_2(x)$ or $b_0(x) \equiv 0$ (see [8], [4]).

Let $g(x) := C^{-1}f(x)$. Substituting

$$(2.5) \quad A(x) = CB(x)C^{-1} \quad \text{and} \quad f(x) = Cg(x)$$

into (2.1) we get

$$(2.6) \quad g(x + y) + g(x - y) = B(y)g(x),$$

that is, with $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$,

$$(2.6a) \quad g_1(x + y) + g_1(x - y) = g_1(x)b_1(y) + g_2(x)b_0(y),$$

$$(2.6b) \quad g_2(x + y) + g_2(x - y) = g_2(x)b_2(y).$$

Substituting also the first of (2.5) into (2.4) we have

$$(2.7) \quad B(u + v) + B(u - v) = B(u)B(v),$$

that is,

$$(2.7a) \quad b_i(u + v) + b_i(u - v) = b_i(u)b_i(v), \quad i = 1, 2,$$

$$(2.7b) \quad b_0(u + v) + b_0(u - v) = b_1(u)b_0(v) + b_0(u)b_2(v).$$

We distinguish three cases.

Case 1. $b_1(x) \equiv b_2(x) \neq 2$. Then $b_1(x_0) = b_2(x_0) \neq 2$ for some $x_0 \in G$. Setting $y = 0$ in (2.6) gives $B(0) = 2I$ (since $g \in \mathcal{F}$) and from (2.7) it follows that $B(2x) + 2I = B(x)^2$. So the matrix $B(2x_0) - 2I = B(x_0)^2 - 4I$ is invertible. This and all subsequent 2×2 matrices are upper triangular with equal diagonal elements and hence commute.

Now the solutions of the matrix equation (2.7) can be written in the form $B(x) = \Xi(x) + \Xi(-x)$ where $\Xi(x) = \chi(x) \begin{pmatrix} 1 & \phi(x) \\ 0 & 1 \end{pmatrix}$ with $\phi \in \mathcal{L}$, $\chi \in \mathcal{M}$ and $\Xi(x + y) = \Xi(x)\Xi(y)$ (see, e.g., the proof of Lemma 1 of [7]).

In order to solve the equation (2.6) we follow the method used in the proof of Theorem 2.2 of [5]. In (2.6) we replace x by $-x$, we subtract the resulting equation from (2.6) and, setting $h(x) = g(x) - g(-x)$, we obtain

$$h(x + y) + h(x - y) = B(y)h(x)$$

with $h(0) = 0$, $h(-x) = -h(x)$ and $h(2x) = B(x)h(x)$. Therefore

$$\begin{aligned} [2B(x + y) - B(x)B(y)]h(x) &= 2B(x + y)h(x) - B(y)h(2x) \\ &= 2[h(2x + y) - h(y)] - [h(2x + y) + h(2x - y)] \\ &= h(y + 2x) + h(y - 2x) - 2h(y) = B(2x)h(y) - 2h(y). \end{aligned}$$

Since $2B(x + y) - B(x)B(y) = [\Xi(x) - \Xi(-x)][\Xi(y) - \Xi(-y)]$ we have

$$[B(2x) - 2I]h(y) = [\Xi(x) - \Xi(-x)][\Xi(y) - \Xi(-y)]h(x).$$

Setting $x = x_0$ we obtain

$$g(y) - g(-y) = [\Xi(y) - \Xi(-y)]\gamma$$

where $\gamma \in K^2$. Adding this to $g(y) + g(-y) = [\Xi(y) + \Xi(-y)]g(0)$ (which follows from (2.6) with $x = 0$) we obtain

$$g(x) = \Xi(x)\alpha + \Xi(-x)\beta,$$

where $\alpha, \beta \in K^2$. So by (2.5) we obtain (2.2) with the first form in (2.3).

Case 2. $b_1(x) \equiv b_2(x) \equiv 2$. Then (2.7b) shows that $b_0 \in \mathcal{N}$ and from (2.6b) (a Jensen equation) we have $g_2(x) = \phi_2(x) + \lambda$ with $\lambda \in K$ and $\phi_2 \in \mathcal{L}$. Now (2.6a) becomes

$$(2.8) \quad g_1(x + y) + g_1(x - y) = 2g_1(x) + [\phi_2(x) + \lambda]b_0(y).$$

If $\phi_2(x) \equiv 0$, then interchanging x, y in (2.8) and subtracting we find

$$g_1(x - y) - g_1(y - x) = [2g_1(x) - \lambda b_0(x)] - [2g_1(y) - \lambda b_0(y)],$$

a Pexider equation which gives $2g_1(x) - \lambda b_0(x) = 2\phi_0(x) + 2\mu$ ($\mu \in K, \phi_0 \in \mathcal{L}$). Hence

$$g(x) = \left\| \begin{array}{c} \frac{1}{2}\lambda b_0(x) + \phi_0(x) + \mu \\ \lambda \end{array} \right\|, \quad B(y) = \left\| \begin{array}{cc} 2 & b_0(y) \\ 0 & 2 \end{array} \right\|.$$

Using (2.5) we obtain (2.2) with the second form in (2.3).

If $\phi_2(x) \not\equiv 0$, then applying Lemma 4 of [1] to (2.8) (here we need the condition $\text{char } K \neq 2, 3$) we obtain

$$g_1(x) = c\phi_2(x)^3 + 3c\lambda\phi_2(x)^2 + \phi_0(x) + \mu, \quad b_0(y) = 6c\phi_2(y)^2,$$

where $c, \mu \in K$ and $\phi_0 \in \mathcal{L}$. So by (2.5) we obtain (2.2) with the third form in (2.3).

Case 3. $b_0(x) \equiv 0$. Then applying Theorem 2.2 of [5] to (2.6a) and (2.6b) we see that $g(x), B(y)$ have one of the forms

$$\begin{aligned} g(x) &= \left\| \begin{array}{c} \lambda\theta(x) + \mu\theta(-x) \\ \nu\eta(x) + \kappa\eta(-x) \end{array} \right\|, & B(y) &= \left\| \begin{array}{cc} \theta(y) + \theta(-y) & 0 \\ 0 & \eta(y) + \eta(-y) \end{array} \right\|, \\ g(x) &= \left\| \begin{array}{c} a_0(x) + \lambda \\ \mu\theta(x) + \nu\theta(-x) \end{array} \right\|, & B(y) &= \left\| \begin{array}{cc} 2 & 0 \\ 0 & \theta(y) + \theta(-y) \end{array} \right\|, \\ g(x) &= \left\| \begin{array}{c} \mu\theta(x) + \nu\theta(-x) \\ a_0(x) + \lambda \end{array} \right\|, & B(y) &= \left\| \begin{array}{cc} \theta(y) + \theta(-y) & 0 \\ 0 & 2 \end{array} \right\|, \\ g(x) &= \left\| \begin{array}{c} a_0(x) + \lambda \\ \phi_0(x) + \mu \end{array} \right\|, & B(y) &= \left\| \begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array} \right\|, \end{aligned}$$

where $\theta, \eta \in \mathcal{M}$, $a_0, \phi_0 \in \mathcal{L}$ and $\kappa, \lambda, \mu, \nu \in K$. So by (2.5) we obtain (2.2) with the last form in (2.3).

To prove the converse observe that the substitution of (2.2) into (2.1) gives

$$\begin{aligned} & [E(x + y) + E(x - y) - [E(y) + E(-y)]E(x)]\alpha \\ & + [E(-x + y) + E(-x - y) - [E(y) + E(-y)]E(-x)]\beta = 0. \end{aligned}$$

On the other hand, it is easy to verify that the functions E in (2.3) satisfy the functional equation

$$(2.9) \quad E(x + y) + E(x - y) = [E(y) + E(-y)]E(x) \quad (x, y \in G)$$

and this completes the proof of Theorem 1. □

In the following theorem the prime indicates transposition.

Theorem 2. *If $f: G \rightarrow K, g, h: G \rightarrow K^2$ is a solution of the functional equation*

$$(2.10) \quad f(x + y) + f(x - y) = h'(y)g(x) \quad (x, y \in G),$$

then f has the form

$$(2.11) \quad f(x) = \lambda'[E(x)\alpha + E(-x)\beta],$$

where $\lambda, \alpha, \beta \in K^2$ and E is given by (2.3).

Conversely, for any f as described above there exist g, h which together with f constitute a solution of (2.10).

Notes. 1. Once f is known, it is easy to determine the complete list for the vector-valued functions g, h .

2. The right-hand side of (2.10), as a scalar product, can also be written $g'(x)h(y)$.

Proof. If $g \notin \mathcal{F}$ or $h \notin \mathcal{F}$, then (2.10) reduces to a Wilson equation. Applying Theorem 2.2 of [5] to this equation we see that the solutions are contained in (2.11). So in the following we assume that $g, h \in \mathcal{F}$.

Setting $y = u + v$ and $y = u - v$ in (2.10), adding the resulting equations and using again (2.10) we find

$$h'(v)[g(x + u) + g(x - u)] = [h'(u + v) + h'(u - v)]g(x).$$

Since $h \in \mathcal{F}$, using Lemma 14.1 of [2] we obtain

$$g(x + u) + g(x - u) = A(u)g(x),$$

with $A: G \rightarrow K_2$. Since $g \in \mathcal{F}$, applying Theorem 1 to this equation we obtain

$$g(x) = C[E(x)\alpha + E(-x)\beta],$$

where $\alpha, \beta \in K^2, C \in K_2$ and E is given by (2.3).

Now from (2.10) with $y = 0$ we have $f(x) = \frac{1}{2}h'(0)g(x)$ which leads to (2.11).

The converse is verified by substitution of (2.11) into (2.10) and using (2.9). \square

Remark. The following result generalizes Theorem 1 and is proved in the same way. Here S is an arbitrary set.

Theorem 3. If $A: G \rightarrow K_2, f: G \times S \rightarrow K^2$ with $f \in \mathcal{F}$ is a solution of the functional equation

$$(2.12) \quad f(x + y, s) + f(x - y, s) = A(y)f(x, s) \quad (x, y \in G, s \in S),$$

then

$$(2.13) \quad \begin{cases} A(y) = C[E(y, s_0) + E(-y, s_0)]C^{-1}, \\ f(x, s) = C[E(x, s)\alpha(s) + E(-x, s)\beta(s)], \end{cases}$$

where $s_0 \in S, C \in K_2$ ($\det C \neq 0$), $\alpha, \beta: S \rightarrow K^2$ are arbitrary functions and $E(x, s)$ has one of the forms

$$(2.14) \quad \begin{aligned} E(x, s) = \chi(x) & \begin{vmatrix} 1 & \phi(x) \\ 0 & 1 \end{vmatrix}, \\ & \begin{vmatrix} 1 & n(x) + \phi_1(x, s) \\ 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} \phi(x) + 1 & c\phi(x)^3 + 3c\phi(x)^2 + \phi_1(x, s) \\ 0 & \phi(x) + 1 \end{vmatrix}, \\ & \begin{vmatrix} \chi_1(x) - [\chi_1(0) - 1][\phi_1(x, s) + 1] & 0 \\ 0 & \chi_2(x) - [\chi_2(0) - 1][\phi_2(x, s) + 1] \end{vmatrix}, \end{aligned}$$

with $c \in K, \phi \in \mathcal{L}, \chi, \chi_1, \chi_2 \in \mathcal{M}, \chi(x) \neq 0, n \in \mathcal{N}$ and $\phi_1, \phi_2: G \times S \rightarrow K$ additive in the first variable.

Conversely, any A, f as described above satisfy (2.12).

(Clearly, for E given by (2.14), the sum $E(y, s) + E(-y, s)$ is independent of the second variable, that is, $E(y, s) + E(-y, s) = E(y, t) + E(-y, t)$ for all $y \in G$ and $s, t \in S$. So A in the first of (2.13) is independent of the choice of s_0 .)

Theorem 3 can be used, e.g., in the solution of the functional equation

$$f(x+t, y-t) + f(x-t, y+t) = g_1(x, y)h_1(t) + g_2(x, y)h_2(t)$$

which generalizes (2.10) (and (3.2) of [6]).

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