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# DIRECTIONAL UNIFORM ROTUNDITY IN SPACES OF ESSENTIALLY BOUNDED VECTOR FUNCTIONS

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Abstract. A formula is given for the directional uniform rotundity modulus of $L_\infty(X)$, where $X$ is a normed space. Then a necessary and sufficient condition is provided for $L_\infty(X)$ to be uniformly rotund in a direction.

## 1. Introduction

Spaces that are uniformly rotund in every direction were defined by A.L. Garkavi [6] to characterize those normed spaces for which every bounded set has at most one Chebyshev center. V. Zizler [11], Day–James–Swaminathan [2], and H. Fakhoury [4] have further studied this generalization of the uniform rotundity property. For more recent investigations on this topic see [3].

In this paper we prove a formula for the directional modulus of rotundity of $L_\infty(X)$, where $X$ is a normed space. As a consequence, we obtain a complete description of the uniform rotundity directions of such a space, thus generalizing to the vector case the corresponding scalar results of R.R. Phelps [9] and V.I. Zizler [11].

Terminology and notation are standard. Let $X$ be a normed space. As usual $B_X(x,r)$ and $S_X(x,r)$ denote the closed ball and the sphere, respectively, with center $x$ and radius $r>0$. We shall also write $B_X = B_X(0,1)$ and $S_X = S_X(0,1)$.

The space $X$ is said to be *uniformly rotund in the direction* $z \in X$ (UR$\rightarrow z$) if the directional modulus of rotundity

$$\delta_X(\rightarrow z,\epsilon) = \inf\left\{1 - \left\|x + \frac{\lambda}{2}z\right\| \ :\ x, x+\lambda z \in B_X,\ \|\lambda z\| \geq \epsilon\right\} \tag{1}$$

is strictly positive for every $0 < \epsilon \leq 2$.

We prove [5] that

(i) If $\|z\| = 1$

$$\delta_X(\rightarrow z,\epsilon) = \inf\left\{1 - \left\|x + \frac{\epsilon}{2}z\right\| \ :\ \|x\| \leq 1, \|x+\epsilon z\| = 1\right\}. \tag{2}$$

(ii) $\delta_X(\rightarrow z,\epsilon)/\epsilon$ is a non-decreasing function on $(0,2]$.

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As a consequence of (ii) we have that $\delta_X(\to z, \epsilon) \le \epsilon/2$.

The function $\delta_X(\to z, \epsilon)$ is continuous with respect to $z$ and $\epsilon$ separately [7], and, as the following result shows, this function is continuous with respect to the two variables together.

**Lemma 1.** *The function*

$$(z, \epsilon) \in X \setminus \{0\} \times [0, 2) \longrightarrow \delta_X(\to z, \epsilon) \in [0, 1)$$

*is continuous.*

*Proof.* Let $(z_n, \epsilon_n), (z, \epsilon) \in X \setminus \{0\} \times [0, 2)$ and $\lim_{n\to\infty} z_n = z$, $\lim_{n\to\infty} \epsilon_n = \epsilon$. Let $\eta > 0$. Then the monotonic property of $\delta_X(\to z, \epsilon)$ with respect to $\epsilon$ and the continuity of $\delta_X(\to z, \epsilon)$ with respect to $z$ guarantee that

$$\delta_X(\to z_n, \epsilon - \eta) \le \delta_X(\to z_n, \epsilon_n) \le \delta_X(\to z_n, \epsilon + \eta),$$

$$\delta_X(\to z, \epsilon \pm \eta) - \eta \le \delta_X(\to z_n, \epsilon \pm \eta) \le \delta_X(\to z, \epsilon \pm \eta) + \eta,$$

if $n$ is sufficiently large. Thus

$$\delta_X(\to z, \epsilon - \eta) - \eta \le \delta_X(\to z_n, \epsilon_n) \le \delta_X(\to z, \epsilon + \eta) + \eta.$$

Since $\delta_X(\to z, \epsilon)$ is continuous with respect to $\epsilon$, we have $\lim_{n\to\infty} \delta_X(\to z_n, \epsilon_n) = \delta_X(\to z, \epsilon)$. □

The space $X$ is said to be *uniformly rotund* (UR) when the modulus of rotundity

$$\delta_X(\epsilon) = \inf\left\{1 - \left\| \frac{x+y}{2} \right\| : x, y \in B_X, \|x - y\| \ge \epsilon \right\}$$

is positive for every $0 < \epsilon \le 2$ [1].

It is clear that $\delta_X(\epsilon) = \inf\{\delta_X(\to z, \epsilon) : z \in S_X\}$. Moreover $\delta_X(\epsilon)$ is an increasing monotonic function such that $\delta_X(\epsilon) \le \epsilon/2$.

Let $(T, \Sigma, \mu)$ be a positive measure space and $X$ be a normed space. The function $x\colon T \to X$ is said to be *simple* if there exist $T_1, \dots, T_n \in \Sigma$ and $x_1, \dots, x_n \in X$ such that $x = \sum_{i=1}^n x_i \chi_{T_i}$, where $\chi_{T_i}$ is the characteristic function of $T_i$. The function $x\colon T \to X$ is defined as *measurable* if, for every finite measurable set $F$, there exists a sequence of simple functions $\{s_n\}_{n\in\mathbb{N}}$ such that $x\chi_F = \lim_{n\to\infty} s_n$ almost everywhere [10]. The set of measurable functions is a linear space. Moreover if $x\colon T \to X$ is measurable and $g\colon X \to \mathbb{R}$ is continuous, then $g \circ x$ is measurable.

For convenience we assume the measure $\mu$ to be complete. Measurable functions are characterized by the following result, whose proof is essentially included in [8, p. 233].

**Lemma 2.** *The function $x\colon T \to X$ is measurable if and only if, for each finite measurable set $F$,*

(i) *$x$ is essentially separably valued on $F$, that is, there is a null set $N$ such that $x(F \setminus N)$ is separable;*
(ii) *$F \cap x^{-1}(G) \in \Sigma$ for each Borel set $G$.*

Define the *essential image* of $x\colon T \to X$ ($\operatorname{ess\,im} x$) as the set of $u \in X$ such that there exists a finite measurable set $F_u$ with $\mu\{t \in F_u : \|z(t) - u\|_X < r\} > 0$ for every $r > 0$.

We use $L_\infty(X)$ to denote the space of measurable equivalence classes of functions $x\colon T \to X$ such that $t \in T \to \|x(t)\|_X$ is essentially bounded. It is a linear space normed by $\|x\| = \operatorname{ess\,sup}_{t\in T}\{\|x(t)\|_X\}$, where ess sup denotes the essential supremum of the function $x$.

To avoid confusion, from now on we shall use $\|\cdot\|$ for the norm in $L_\infty(X)$ and $\|\cdot\|_X$ for the norm in $X$.

## 2. Results

The main result is a formula for the directional rotundity modulus of $L_\infty(X)$.

**Theorem 3.** *Let $z \in S_{L_\infty(X)}$. Then*

$$\delta_{L_\infty(X)}(\to z, \epsilon) = \operatorname*{ess\,inf}_{t\in T}\{\delta_X(\to z(t), \epsilon\|z(t)\|_X)\}, \quad 0 \le \epsilon < 2, \tag{3}$$

*where* ess inf *denotes the essential infimum.*

*Proof.* Our first goal is to show that

$$\delta_{L_\infty(X)}(\to z, \epsilon) \ge \operatorname*{ess\,inf}_{t\in T}\{\delta_X(\to z(t), \epsilon\|z(t)\|_X)\}, \quad 0 \le \epsilon < 2. \tag{4}$$

Take $x \in L_\infty(X)$ such that $\|x\| \le 1$, $\|x + \epsilon z\| = 1$. Then

$$\|x(t)\|_X \le 1, \qquad \|x(t) + \epsilon z(t)\|_X \le 1,$$

and

$$\|x(t) + (\epsilon/2)z(t)\|_X \le 1 - \delta_X(\to z(t), \epsilon\|z(t)\|_X) \quad \text{almost everywhere.}$$

Therefore

$$\|x + (\epsilon/2)z\| \le 1 - \operatorname*{ess\,inf}_{t\in T} \delta_X(\to z(t), \epsilon\|z(t)\|_X),$$

and (4) holds.

Now for the harder part. Suppose that

$$\delta_{L_\infty(X)}(\to z, \epsilon) > \operatorname*{ess\,inf}_{t\in T}\{\delta_X(\to z(t), \epsilon\|z(t)\|_X)\}.$$

Then there exists $E \in \Sigma$ with $\mu(E) > 0$ such that

$$\delta_X(\to z(t), \epsilon\|z(t)\|_X) < \delta_{L_\infty(X)}(\to z, \epsilon) \quad \text{for every } t \in E. \tag{5}$$

We consider two cases.

*Case* A. There exists $F \subset E$ such that $F \in \Sigma$ and $0 < \mu(F) < \infty$.

To arrive at a contradiction with (5), we prove that

$$\inf\{\delta_X(\to u, \epsilon\|u\|_X) : u \in \operatorname{ess\,im} z\} \le \operatorname*{ess\,inf}_{t\in F}\{\delta_X(\to z(t), \epsilon\|z(t)\|_X)\} \tag{6}$$

and

$$\delta_{L_\infty(X)}(\to z, \epsilon) \le \inf\{\delta_X(\to u, \epsilon\|u\|_X) : u \in \operatorname{ess\,im} z\}. \tag{7}$$

First note that $\operatorname{ess\,im} z \ne \emptyset$. To prove this, suppose that $\operatorname{ess\,im} z = \emptyset$. Since $F$ has a finite measure, there is a null set $N$ and a dense sequence $\{u_n\}_{n\in\mathbb{N}}$ in $z(F\setminus N)$. Clearly $u_n \notin \operatorname{ess\,im} z$. Then there exists $r_n > 0$ such that $\mu\{z^{-1}(B(u_n, r_n)) \cap F\} = 0$. Moreover $F\setminus N \subset \bigcup_{n\in\mathbb{N}} z^{-1}(B(u_n, r_n))$, and then $\mu(F\setminus N) = 0$. This contradicts the hypothesis $\mu(F) > 0$.

To show (6), we verify that $\alpha := \inf\{\delta_X(\to u, \epsilon\|u\|_X) : u \in \operatorname{ess\,im} z\}$ is an essential lower bound in $F$ of the function $\delta_X(\to z(\cdot), \epsilon\|z(\cdot)\|_X)$. That is, we must prove that $\mu\{t \in F : \delta_X(\to z(t), \epsilon\|z(t)\|_X) < \alpha\} = 0$. Define

$$u \in X \setminus \{0\} \to g(u) := \delta_X(\to u, \epsilon\|u\|_X).$$

Then

$$\mu\{t \in F : \delta_X(\to z(t), \epsilon\|z(t)\|_X) < \alpha\} = \mu\{(F \setminus N) \cap (g \circ z)^{-1}(-\infty, \alpha)\}.$$

If $(F \setminus N) \cap (g \circ z)^{-1}(-\infty, \alpha) = \emptyset$, then $\mu\{(F \setminus N) \cap (g \circ z)^{-1}(-\infty, \alpha)\} = 0$. Otherwise, since by Lemma 2, $z(F \setminus N) \cap \big(g^{-1}(-\infty, \alpha)\big)$ is separable, it contains a dense sequence $\{w_n\}_{n \in \mathbb{N}}$. Clearly $z(t) \notin \operatorname{ess\,im} z$ if $t \in (F \setminus N) \bigcap (g \circ z)^{-1}(-\infty, \alpha)$. Hence, there is some $s_n > 0$ such that $\mu\{z^{-1}(B(w_n, s_n)) \cap F\} = 0$. From the density of the sequence $\{w_n\}_{n \in \mathbb{N}}$ we have

$$\mu\{(F \setminus N) \cap (g \circ z)^{-1}(-\infty, \alpha)\} \le \sum_{n=1}^{\infty} \mu\{z^{-1}(B(w_n, s_n)) \cap F\} = 0.$$

Now we prove (7). Let $u \in \operatorname{ess\,im} z$. From the definition of $\operatorname{ess\,im} z$ it follows that $\|u\|_X \le \|z\| = 1$. Take $a \in B_X$ such that $a + \epsilon u \in S_X$, and define $T_n = z^{-1}(B(u, 1/n)) \cap F_u$ for every $n \in \mathbb{N}$. Then $T_n \in \Sigma$ and $\mu(T_n) > 0$. Set

$$\begin{aligned} x_n(t) &= a\chi_{T_n}(t) - z(t)\chi_{T \setminus T_n}(t) \quad \text{for } t \in T, \\ z_n(t) &= u\chi_{T_n}(t) + z(t)\chi_{T \setminus T_n}(t) \quad \text{for } t \in T. \end{aligned}$$

Both functions are measurable and belong to the unit ball of $L_\infty(X)$. Moreover the functions

$$\begin{aligned} x_n(t) + \epsilon z_n(t) &= (a + \epsilon u)\chi_{T_n}(t) + (\epsilon - 1)z(t)\chi_{T \setminus T_n}(t) \quad \text{for } t \in T, \\ x_n(t) + \frac{\epsilon}{2} z_n(t) &= \Big(a + \frac{\epsilon}{2} u\Big)\chi_{T_n}(t) + \Big(\frac{\epsilon}{2} - 1\Big) z(t)\chi_{T \setminus T_n}(t) \quad \text{for } t \in T, \end{aligned}$$

satisfy $\|x_n + \epsilon z_n\| = 1$ and $\|x_n + (\epsilon/2)z_n\| \ge \|a + (\epsilon/2)u\|_X$. Hence

$$\delta_X(\to u, \epsilon\|u\|_X) \ge \delta_{L_\infty(X)}(\to z_n, \epsilon\|z_n\|_X).$$

Since

$$\|z(t) - z_n(t)\|_X = \|z(t) - u\|_X \chi_{T_n}(t) < \frac{1}{n},$$

we have $\lim_{n \to \infty} \|z - z_n\| = 0$. Then by Lemma 1

$$\lim_{n \to \infty} \delta_{L_\infty(X)}(\to z_n, \epsilon\|z_n\|) = \delta_{L_\infty(X)}(\to z, \epsilon\|z\|),$$

which completes the proof of (7).

*Case* B. For every $F \subset E$ such that $F \in \Sigma$, either $\mu(F) = 0$ or $\mu(F) = +\infty$.

We show that

$$\delta_{L_\infty(X)}(\to z, \epsilon) \le \operatorname*{ess\,inf}_{t \in E}\{\delta_X(\to z(t), \epsilon\|z(t)\|_X)\} := \beta, \tag{8}$$

which contradicts (5).

Let $r > 0$ and $F = \{t \in E : \delta_X(\to z(t), \epsilon\|z(t)\|_X) \le \beta + r\}$. For every $t \in F$ except for a null measurable set, we can choose $y(t) \in B_X$ and $y(t) + \epsilon z(t) \in S_X$ such that

$$\delta_X(\to z(t), \epsilon\|z(t)\|_X) \le 1 - \Big\|y(t) + \frac{\epsilon}{2} z(t)\Big\|_X \le \beta + r.$$

Define $x(t) = y(t)\chi_F(t) - (\epsilon/2)z(t)\chi_{T \setminus F}(t)$. Then

$$x(t) + \epsilon z(t) = (y(t) + \epsilon z(t))\chi_F(t) + (\epsilon/2)z(t)\chi_{T \setminus F}(t).$$

By Lemma 2 both functions are measurable, since in every finite measurable set $x(t) = -\epsilon z(t)/2$ and $x(t) + \epsilon z(t) = \epsilon z(t)/2$ almost everywhere. Also, $x, x + \epsilon z \in B_{L_\infty(X)}$, and $\|x + (\epsilon/2)z\| = \operatorname{ess\,sup}_{t\in F}\{\|y(t) + (\epsilon/2)z(t)\|_X\}$. Then

$$\delta_{L_\infty(X)}(\to z, \epsilon) \leq 1 - \left\|x + \frac{\epsilon}{2}z\right\| \leq \beta + r.$$

As $r > 0$ is an arbitrary number, (8) holds. □

**Corollary 4.** *The rotundity modulus of $L_\infty := L_\infty(\mathbb{R})$ in the direction $\zeta \in S_{L_\infty}$ is*

$$\delta_{L_\infty}(\to \zeta, \epsilon) = \frac{\epsilon}{2}\operatorname{ess\,inf}\{|\zeta|\}, \quad 0 \leq \epsilon < 2. \tag{9}$$

*Proof.* When the dimension of $X$ is equal to 1, $\delta_X(\epsilon) = \epsilon/2$. □

Next we provide a complete description of the uniform rotundity directions in the space $L_\infty(X)$.

**Theorem 5.** *Let $X$ be a normed space.*

(i) *The space $L_\infty(X)$ is* UR $\to z$, $z \in S_{L_\infty(X)}$, *if and only if*

$$\operatorname*{ess\,inf}_{t\in T}\{\delta_X(\to z(t), \epsilon\|z(t)\|_X)\} > 0 \quad \text{for } 0 < \epsilon \leq 2.$$

(ii) *Let $X$ be a* UR *normed space. Then $L_\infty(X)$ is* UR $\to z$ *if and only if*

$$\operatorname*{ess\,inf}_{t\in T}\{\|z(t)\|_X\} > 0.$$

*Proof.* Part (i) is trivial after Theorem 3. To prove part (ii) we may assume that $z \in S_{L_\infty(X)}$. If $L_\infty(X)$ is UR $\to z$ then from $\delta_X(\to z(t), \epsilon\|z(t)\|_X) \leq (\epsilon/2)\|z(t)\|_X$ almost everywhere, we have that $\operatorname{ess\,inf}_{t\in T}\{\|z(t)\|_X\} > 0$. For the converse, note that $\delta_X(\epsilon\operatorname{ess\,inf}_{t\in T}\{\|z(t)\|_X\}) \leq \delta_X(\epsilon\|z(t)\|_X) \leq \delta_X(\to z(t), \epsilon\|z(t)\|_X)$. □

When $(T, \Sigma, \mu)$ is a discrete measure space, one has $L_\infty(X) = \ell_\infty(X)$ and $\operatorname{ess\,inf} = \inf$. Then Theorems 3 and 5 and Corollary 4 hold for $\ell_\infty(X)$, although a simpler proof of Theorem 3 is available since every function is measurable in this case. Moreover formula (3) also holds at $\epsilon = 2$. The same results can be obtained for $\ell_\infty(X_i)$, where $\{X_i\}_{i\in I}$ is a family of normed spaces, i.e., for the space of functions $x\colon I \to \bigcup_{i\in I} X_i$, such that $x_i \in X_i$, for each $i \in I$, and $(\|x_i\|_i) \in \ell_\infty$, which is a linear space endowed with the norm $\|x\| = \sup_{i\in I}\|x_i\|_i$.

As an application of the previous results, we show that the directional modulus of rotundity is not continuous at $\epsilon = 2$.

**Example 6.** For each $i = 1, 2\ldots$ let $X_i$ be the space $\mathbb{R}^2$ endowed with the norm

$$\|(r, s)\|_{i+1} = \left(|r|^{i+1} + |s|^{i+1}\right)^{\frac{1}{i+1}}.$$

Let $z = (z_i)$, where $z_i = (1, 0)$. Then

$$\delta_{\ell_\infty(X_i)}(\to z, \epsilon) = \inf\{\delta_{X_i}(\to z_i, \epsilon) : i = 1, 2\ldots\} = 0 \quad \text{for } 0 \leq \epsilon < 2$$

and $\delta_{\ell_\infty(X_i)}(\to z, 2) = 1$.

## References


[1] J.A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

[2] M.M. Day, R.C. James, S. Swaminathan, *Normed linear spaces that are uniformly convex in every direction*, Canad. J. Math. Vol. **26** (6) (1971) , 1051–1059. MR **44:**4492

[3] R. Deville, G. Godefroy, V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 64, Longman Scientific & Tecnichal, 1993. MR **94d:**46012

[4] H. Fakhoury, *Directions d'uniform convexité dans un space normé*, Séminaire Choquet, 14 anné, n. **6** (1974). MR **57:**17223

[5] M. Fernandez, I. Palacios, *Relative rotundity in* $L^p(X)$, Arch. Math. (Basel) **65** (1995), 61–68. CMP 95:14

[6] A.L. Garkavi, *The best possible net and the best possible cross-section of a set in a normed space*, Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962), 87–106; Amer. Math. Soc. Transl. Ser. 2, **39** (1964), 111–132. MR **25:**429

[7] A. Kamińska, B. Turett, *Some remarks on moduli of rotundity in Banach spaces*, Acad. Scien. Math. Vol. **36**, No. 5–6, (1988). MR **92b:**46020

[8] S. Lang, *Real Analysis*, Addison–Wesley Pub. Comp., (1969).

[9] R.R. Phelps, *Uniqueness of Hahn–Banach extensions and unique best approximation*, Trans. Amer. Math. Soc. **95** (1960), 238–255. MR **22:**3964

[10] A.C. Zaanen, *Integration*, North-Holland Publ. Com., Amsterdam, (1967). MR **36:**5286

[11] V. Zizler, *On some rotundity and smoothness properties in Banach spaces*, Dissertationes Math. **87** (1971), 1–37. MR **45:**9108



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