PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 5, May 1997, Pages 1377–1379
S 0002-9939(97)03767-2

PEAK SET WITHOUT PEAK POINTS

KRZYSZTOF JAROSZ

(Communicated by Theodore W. Gamelin)

Abstract. We give an example of a natural Banach function algebra on the unit disc such that a smaller disc is a peak set for the algebra, but it does not contain any peak point.

A Banach function algebra on a compact Hausdorff space X is a Banach algebra A consisting of continuous functions on X, such that A separates points of X and contains the constant functions. If the norm of the algebra A coincides with the sup norm on X, it is called a uniform algebra. If any linear and multiplicative functional on A is of the form $f \mapsto f(x)$ for some $x \in X$, the algebra is called natural. A subset K of X is a peak set for A if there is an $f \in A$ such that $f \equiv 1$ on K and $|f(x)| < 1$ for $x \notin K$; if $K = \{x_0\}$, then x_0 is a peak point. If no proper subset of K is a peak set we call it a minimal peak set for A. It is well known [2] that if A is a uniform algebra on a metrizable set X then any peak set contains a peak point. In [1] H. G. Dales constructed a natural Banach function algebra on a compact subset of $\mathbb{C}^2$ having a peak set without any peak point. T. G. Honary [3] provided an example in $\mathbb{R}^3$, however his algebra is not natural. In this note we give a very simple example of a natural Banach function algebra on the unit disc with peak sets not containing any peak point.

Let K be an open nonempty subset of the complex plane $\mathbb{C}$. By $C^1(K)$ we denote the algebra of all bounded complex valued functions on $\overline{K}$ with continuous and bounded first order partial derivatives on K. $C^1(K)$ is a Banach function algebra on $\overline{K}$ if equipped with the norm

$$\|f\| = \|f\|_\infty + \|f_x\|_\infty + \|f_y\|_\infty ,$$

where $\|\cdot\|_\infty$ is the sup norm. By $A(K)$ we denote the uniform algebra of all continuous functions on $\overline{K}$ which are analytic on K. We put $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{C}_- = \{z : \operatorname{Re} z < 0\}$.

Theorem 1. *Let A be a subalgebra of $C^1(\mathbb{D}_1)$ consisting of all functions which are analytic on $\mathbb{D}_{\frac{1}{2}}$. Then $\overline{\mathbb{D}_{\frac{1}{2}}}$ is a minimal peak set for A; in particular $\overline{\mathbb{D}_{\frac{1}{2}}}$ does not contain any peak point.*

Lemma 2. *There is no $h \in A(\mathbb{D}_1)$ such that $h(1) = 0$ and $|1 + (z-1)\,h(z)| < 1$ for $z \neq 1$.*

Proof of the lemma. Assume that such a function does exist. If we compose $1 + (z-1)\,h(z)$ with a suitable fractional linear transformation we get an $f \in A(\mathbb{C}_-)$

Received by the editors November 8, 1995.
1991 *Mathematics Subject Classification.* Primary 46J10.

such that $f(z) = 1 + zg(z)$, $g(0) = 0$, and $|f(z)| < 1$ for $z \in \overline{\mathbb{C}_-} \backslash \{0\}$. Let α be a positively oriented curve consisting of the segment from $-i$ to $+i$ and the half of the unit circle contained in $\mathbb{C}_-$. We shall show that $\gamma = g \circ \alpha$ has negative orientation, which will contradict the fact that g is analytic.

For any $z \in \overline{\mathbb{C}_-} \backslash \{0\}$ we have $\operatorname{Re} zg(z) < 0$, so

- as z moves up from 0 to $+i$ along the curve α, $\operatorname{Im} g(z) > 0$,
- as z moves along the half circle of α, $g(z)$ is in $\mathbb{C} \backslash \{z : \operatorname{Re} z \leq 0, \ \operatorname{Im} z = 0\}$,
- as z moves from $-i$ to 0 on the vertical line, $\operatorname{Im} g(z) < 0$.

Consequently, if z_0 is a positive real number such that the segment $(0, z_0]$ does not intersect γ, then the orientation of γ around z_0, as well as around any point from $(0, z_0]$, is negative. □

Proof of the theorem. It is well known that there is a C^1-function, and even a C^∞-function on $\mathbb{D}_1$, peaking exactly on $\overline{\mathbb{D}_{\frac{1}{2}}}$; thus $\overline{\mathbb{D}_{\frac{1}{2}}}$ is a peak set.

Assume $\frac{1}{2}$ is a peak point for A and let $f \in A$ be a function peaking at $\frac{1}{2}$. Put $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. Since $u^2 + v^2$ has the maximum at $\frac{1}{2}$, and $u\left(\frac{1}{2}\right) = 1$, $v\left(\frac{1}{2}\right) = 0$, it follows that $u_x\left(\frac{1}{2}\right) = u_y\left(\frac{1}{2}\right) = 0$. Since the partial derivatives are continuous, and f satisfies the Cauchy-Riemann equations on $\mathbb{D}_{\frac{1}{2}}$, we get $v_x\left(\frac{1}{2}\right) = v_y\left(\frac{1}{2}\right) = 0$. Hence

$$g(z) \stackrel{df}{=} \frac{f(z) - 1}{z - \frac{1}{2}} \to 0, \qquad \text{as } z \to \frac{1}{2},$$

so $g \in A\left(\mathbb{D}_{\frac{1}{2}}\right)$, $g\left(\frac{1}{2}\right) = 0$, and $f(z) = 1 + \left(z - \frac{1}{2}\right) g(z)$ has a strict maximum at $\frac{1}{2}$. This contradicts the lemma. Hence $\overline{\mathbb{D}_{\frac{1}{2}}}$ does not contain any peak point.

What remains to show is that $\overline{\mathbb{D}_{\frac{1}{2}}}$ does not contain any proper peak set. Assume $K \subsetneq \overline{\mathbb{D}_{\frac{1}{2}}}$ is a peak set and let $F \in A$ be a corresponding function peaking on K. Notice that K cannot contain the entire circle $\partial\mathbb{D}_{\frac{1}{2}}$. Let $z_0 \in K \cap \partial\mathbb{D}_{\frac{1}{2}}$ and $z_1 \in \partial\mathbb{D}_{\frac{1}{2}} \backslash K$. For any $w \in \partial\mathbb{D}_{\frac{1}{2}} \backslash \{z_0\}$ let φ_w be a C^1-automorphism of $\overline{\mathbb{D}}$ mapping $\overline{\mathbb{D}_{\frac{1}{2}}}$ onto itself, analytic on $\mathbb{D}_{\frac{1}{2}}$, and such that $\varphi_w(z_0) = z_0$ and $\varphi_w(w) = z_1$. Such an automorphism can be obtained by smoothly extending a suitable fractional linear automorphism of $\mathbb{D}_{\frac{1}{2}}$. Sets of the form $U_w = \left\{z \in \overline{\mathbb{D}_{\frac{1}{2}}} \backslash \{z_0\} : |F \circ \varphi_w(z)| < 1\right\}$ form an open cover of $\partial\mathbb{D}_{\frac{1}{2}} \backslash \{z_0\}$. Let $\{w_1, w_2, ...\}$ be such that $\bigcup_{j=1}^{\infty} U_{w_j} = \partial\mathbb{D}_{\frac{1}{2}} \backslash \{z_0\}$. Put $G = \sum_{j=1}^{\infty} \frac{F \circ \varphi_{w_j}}{2^j \left\| F \circ \varphi_{w_j} \right\|}$. This function G belongs to A and peaks exactly at z_0, which contradicts the previous part of the proof. □

Notice that essentially the same proof can be repeated to provide a slightly more general example:

Let Ω_j, $j = 1, 2, ...$, be a sequence of open subsets of $\mathbb{D}_1$ such that for any j, $\overline{\Omega_j}$ has an open neighborhood contained in $\mathbb{D}_1$ not intersecting $\bigcup_{i \neq j} \Omega_i$, and such that $\partial\Omega_j$ is a union of finitely many smooth Jordan curves. Let A be a subalgebra of $C^1(\mathbb{D}_1)$ consisting of all functions which are analytic on $\bigcup \Omega_j$. Then for any j, $\overline{\Omega_j}$ is a peak set not containing any peak point.

However, it is not clear whether the result can be further extended to Swiss cheese type domains to provide an example of a natural Banach function algebra where the union of all nontrivial minimal peak sets is dense in the unit disc. The

difficulty, in general, is not in proving that Ω_j does not contain any peak point, but in proving that $\overline{\Omega_j}$ is a peak set.

References

[1] H. G. Dales. Boundaries and peak points for Banach function algebras. *Proc. London Math. Soc.*, 22(3):121–136, 1971. MR **43:**2510

[2] T. W. Gamelin. *Uniform Algebras*. Chelsea Pub. Comp., New York, 1984. MR **53:**14137

[3] T. G. Honary. An example of a Banach function algebra having a peak set without any peak point. In *Proceedings of The Fifth Analysis Seminar at Shiraz University*, pages 26–32, 1990.

Department of Mathematics, Bowling Green State University, Bowling Green, Ohio 43403

Current address: Department of Mathematics & Statistics, Southern Illinois University, Edwardsville, Illinois 62026

E-mail address: kjarosz@siue.edu