

NONSTANDARD MODELS AND ANALYTIC EQUIVALENCE RELATIONS

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ABSTRACT. We improve a result of Hjorth concerning the nature of thin analytic equivalence relations. The key lemma uses a weakly compact cardinal to construct certain nonstandard models, which Hjorth obtained using $\#$'s.

The purpose of this note is to improve the following result of Hjorth [2].

Theorem (Hjorth). *Suppose that for every real x , $x^\#$ exists. Let E be an analytic equivalence relation, Σ_1^1 in parameter x_0 . Then either there exists a perfect set of pairwise E -inequivalent reals or every E -equivalence class has a representative in a set-generic extension of $L[x_0]$.*

Hjorth's proof makes use of his analysis of nonstandard Ehrenfeucht-Mostowski models built from $\#$'s. Instead, we construct the necessary nonstandard models using infinitary model theory, assuming only the existence of weak compacts.

Theorem 1. *Suppose that for every real x there is a weakly compact cardinal in $L[x]$. Then the conclusion of Hjorth's Theorem still holds.*

The main lemma is the following.

Lemma 2. *Suppose that there is a weakly compact cardinal in $L[x]$, x a real. Then there is a countable nonstandard ω -model M_x of ZF such that $x \in M_x$ and $L(M_x) = L$ in the sense of M_x has an isomorphic copy in a set-generic extension of $L[x_0]$, for any real x_0 .*

It is not known if the conclusion of Lemma 2 holds in ZFC alone, for arbitrary x (with ZF replaced by an arbitrary finite subtheory).

Proof of Theorem 1 from Lemma 2 (as in Hjorth [2]). Suppose that E is an analytic equivalence relation, Σ_1^1 in the parameter x_0 , and choose an x_0 -recursive tree T on $\omega \times \omega \times \omega^\omega$ such that $xEy \iff T(x, y)$ has a branch. For each countable ordinal α we define $xE_\alpha y \iff \text{rank}(T(x, y))$ is at least α ; then E_α is Borel in (x_0, c) where c is any real coding α and E is the intersection of the E_α 's. We may assume that each E_α is an equivalence relation (see Theorem 1.4 of Hjorth [2]). A theorem of Harrington and Silver says that a Π_1^1 -equivalence relation has a perfect set of pairwise inequivalent reals or each equivalence class has a representative constructible from the parameter defining the relation. As E_α is Borel in (x_0, c) where c is a

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real coding α , and as we may assume that E and hence each E_α has no perfect set of pairwise inequivalent reals, we know that each E_α -equivalence class has a representative in $L[x_0, c]$ where c is any real coding α .

Now let x be arbitrary and by Lemma 2 choose a countable nonstandard ω -model M_x of ZF containing (x_0, x) such that $L(M_x)$ has an isomorphic copy in a set-generic extension N of $L[x_0]$. Let $a \in \text{ORD}(M_x)$ be nonstandard and let c be a code for a , generic over M_x ; then by applying Harrington-Silver in $M_x[c]$ we conclude that there is y in $L(M_x)[x_0, c]$ such that $yE_a x$. By choosing c to be generic over N as well we get that y belongs to a set-generic extension of $L[x_0]$. Finally, $yE x$ since, if not, $yE_\alpha x$ would fail for some α admissible in (y, x) and hence for some (standard) $\alpha < a$. \dashv

To prove Lemma 2 we discuss infinitary logic. Fix a real x and assume $V = L[x]$. Let κ be weakly compact and introduce the language \mathcal{L} consisting of the formulas in the language of set theory with constants \underline{a} for $a \in L_\kappa[x]$, closed under conjunctions and disjunctions of size less than κ (however we allow a formula to have only finitely many free variables). Let T be the theory of $\langle L_\kappa[x], a \rangle$, $a \in L_\kappa[x]$, in this language. An n -type is a set of formulas Γ with free variables $v_1 \dots v_n$, and such a Γ is *consistent with T* if there is a model M of T and $m_1 \dots m_n$ in M such that $M \models \varphi(m_1 \dots m_n)$ for each $\varphi \in \Gamma$, where M exists in a set-generic extension of $V = L[x]$. Γ is *complete* if for every $\varphi(v_1 \dots v_n)$ either φ or $\sim \varphi$ belongs to Γ .

Now work in the Lévy collapse $L[x, c]$, where c is a real coding κ^+ of $L[x]$. Let d_1, d_2, \dots be ω -many new constant symbols and for $D \subseteq \{d_1, d_2, \dots\}$ let the language \mathcal{L}_D be defined like \mathcal{L} but with the new constant symbols from D . Define $T_0 = T \subseteq T_1 \subseteq \dots$ and $D_0 = \emptyset \subseteq D_1 \subseteq D_2 \subseteq \dots$ inductively as follows: if T_n, D_n have been defined select a complete k -type $\Gamma_n(\vec{v})$ in $L[x]$ consistent with T_n , choose $D_n \subseteq D_{n+1}$ so that $\text{card}(D_{n+1} - D_n) = k$, and let $T_{n+1} = T_n \cup \Gamma_n(\vec{d})$, where \vec{d} enumerates $D_{n+1} - D_n$. This can be done in such a way that $\bigcup_n T_n = T^*$ is $L[x]$ -saturated: if $\Gamma(\vec{v}, \vec{w})$ is an $L[x]$ -type, \vec{d} a finite sequence from D , and $\Gamma(\vec{d}, \vec{w})$ consistent with T^* , then T^* includes $\Gamma(\vec{d}, \vec{e})$ for some \vec{e} . And note that each T_n belongs to $L[x]$ (though of course T^* itself makes use of the Lévy collapse c).

Let M_x be the model determined by T^* , whose universe consists of (equivalence classes of) the constants $d_n, n \in \omega$. Note that a set in $L[x]$ of sentences in some \mathcal{L}_D is consistent iff each subset of $L[x]$ -cardinality $< \kappa$ is, by Π_1^1 -reflection and the equivalence of consistency with existence of a model after performing a Lévy collapse of κ . An easy consequence is that M_x is nonstandard with standard ordinal κ .

Now consider $L(M_x)$: every n -type in the language $\mathcal{L}_0 =$ (same as \mathcal{L} but restricted to L_κ) that is realized in $L(M_x)$ belongs to L , as each of its initial segments (obtained by restricting to some $L_\alpha, \alpha < \kappa$) belongs to L and κ is weakly compact. Also, just as M_x is saturated for types in $L[x]$, $L(M_x)$ is saturated for types in L , since again by weak compactness any \mathcal{L}_0 -type in L consistent with T can be extended to a complete \mathcal{L} -type consistent with T in $L[x]$.

Now it is clear that $L(M_x)$ has an isomorphic copy in $L[c]$: using c we can do the same construction as we did above in $L[x, c]$, obtaining M_0 , a model that is saturated for \mathcal{L}_0 -types in L and realizing only types in L . Now construct an isomorphism via a back and forth argument in ω steps between M_0 and $L(M_x)$.

Finally, note that by absoluteness the desired model M_x exists not only in $L[x, c]$ but in $L[x]$. \dashv

Remark. Lemma 2 can also be used to establish the following improvement of the Glimm-Effros style dichotomy theorem of Hjorth and Kechris [3]: Let E be a Σ_1^1 equivalence relation. Assume that for every real x there is a weakly compact cardinal in $L[x]$. Then either E_0 is continuously reducible to E , or E is reducible to $2^{<\omega_1}$ by a function Δ_2^1 in the codes.

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