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UPPER BOUNDS FOR THE NUMBER OF FACETS OF A SIMPLICIAL COMPLEX

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Abstract. Here we study the maximal dimension of the annihilator ideals $0 :_A m^j$ of artinian graded rings $A = P/(I, x_1^2, x_2^2, \dots, x_v^2)$ with a given Hilbert function, where P is the polynomial ring in the variables $x_1, x_2, \dots, x_v$ over a field K with each $\deg x_i = 1$, I is a graded ideal of P, and m is the graded maximal ideal of A. As an application to combinatorics, we introduce the notion of j-facets and obtain some informations on the number of j-facets of simplicial complexes with a given f-vector.

Let $P = K[x_1, x_2, \dots, x_v]$ denote the polynomial ring in v variables over a field K with the standard grading, i.e., each $\deg x_i = 1$, and write $K\{\Gamma\}$ for the quotient algebra $P/(x_1^2, x_2^2, \dots, x_v^2)$. We are interested in the dimensions of the annihilator ideals $0 :_{K\{\Gamma\}/I} m^j$ of $K\{\Gamma\}/I$, where m is the graded maximal ideal of $K\{\Gamma\}/I$. In particular, among all graded ideals I of $K\{\Gamma\}$ with a given Hilbert function, we determine the maximal dimension of the socles $0 :_{K\{\Gamma\}/I} m$ of $K\{\Gamma\}/I$. The graded ring $K\{\Gamma\}/I$ is studied in [A–H–H] when I is generated by (squarefree) monomials.

First, we recall some standard notation and terminology on graded rings and modules. When M is a $\mathbf{Z}$-graded module, where $\mathbf{Z}$ is the set of integers, we write M_i, $i \in \mathbf{Z}$, for the i-th graded component of M. Moreover, for every $a \in \mathbf{Z}$, we define $M(a)$ to be the $\mathbf{Z}$-graded module with graded components $M(a)_i = M_{a+i}$ for all $i \in \mathbf{Z}$. If M is a finitely generated $\mathbf{Z}$-graded module over the polynomial ring $P = K[x_1, x_2, \dots, x_v]$, then the modules $\operatorname{Tor}_i^K(K, M)$ are finite-dimensional graded K-vector spaces. Then we say that $\beta_{ij}(M) := \dim_K \operatorname{Tor}_i^K(K, M)_j$ is the (i, j)-th graded Betti number of M. Finally, when A is a graded ring over K and J is a graded ideal of A, we denote by $0 :_A J$ the annihilator of J in A.

Let $\binom{V}{q}$ denote the set of all squarefree monomials of degree $q \geq 1$ in the variables $V = \{x_1, x_2, \dots, x_v\}$. We write $\leq_{\text{lex}}$ for the lexicographic order on $\binom{V}{q}$, i.e., if $S = x_{i_1} x_{i_2} \cdots x_{i_q}$ and $T = x_{j_1} x_{j_2} \cdots x_{j_q}$ are squarefree monomials belonging to $\binom{V}{q}$ with $1 \leq i_1 < i_2 < \cdots < i_q \leq v$ and $1 \leq j_1 < j_2 < \cdots < j_q \leq v$, then $S <_{\text{lex}} T$ if $i_1 = j_1, \dots, i_{k-1} = j_{k-1}$ and $i_k > j_k$ for some $1 \leq k \leq q$. A nonempty set $\mathcal{M} \subset \binom{V}{q}$ is called a *squarefree lexsegment set of degree* q if $T \in \mathcal{M}$, $S \in \binom{V}{q}$ and $T \leq_{\text{lex}} S$ imply $S \in \mathcal{M}$. An ideal I of $K\{\Gamma\}$ generated by squarefree monomials is

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called a *squarefree lexsegment ideal* if, for every $1 \leq q \leq v$, $T \in I \cap \binom{V}{q}$, $S \in \binom{V}{q}$ and $T <_{\text{lex}} S$ imply $S \in I$.

We are now in the position to state our algebraic result of this paper.

Theorem. (a) *Suppose that I is a graded ideal of $K\{\Gamma\}$ with $I_0 = I_1 = (0)$. Then, there exists a unique squarefree lexsegment ideal I^{lex} of $K\{\Gamma\}$ with the same Hilbert function as I.*

(b) *Let m be the graded maximal ideal of $K\{\Gamma\}$. Fix $j \geq 0$. Suppose that for every $i \geq 0$ we have*

$$\dim_K(I/m^j I)_i = \dim_K(I^{\text{lex}}/m^j I^{\text{lex}})_i.$$

Then for every $i \geq 0$ we have

$$\dim_K(I/m^{j+1} I)_i \leq \dim_K(I^{\text{lex}}/m^{j+1} I^{\text{lex}})_i.$$

(c) *Let $A = K\{\Gamma\}/I$ and $B = K\{\Gamma\}/I^{\text{lex}}$. Fix $j \geq 0$. Suppose that for every $i \geq 0$ we have*

$$\dim_K(0 :_A m^j)_i = \dim_K(0 :_B m^j)_i.$$

Then for every $i \geq 0$ we have

$$\dim_K(0 :_A m^{j+1})_i \leq \dim_K(0 :_B m^{j+1})_i.$$

Proof. First, we choose any term order ρ for the monomials in the polynomial ring $P = K[x_1, x_2, \dots, x_v]$ and we write $J \subset P$ for the preimage of I under the canonical epimorphism $P \to K\{\Gamma\}$. It is well known (e.g., [M–M], [B–H–V]) that P/J and $P/\text{in}_\rho(J)$ have the same Hilbert function and that we have the inequality $\beta_{ij}(P/J) \leq \beta_{ij}(P/\text{in}_\rho(J))$ for every i and j.

We have the equalities $\beta_{1j}(P/J) = \dim_K(I/mI)_j$ if $j > 2$; $\beta_{12}(P/J) = \dim_K(I/mI)_2 - v$; and $\beta_{vj}(P/J) = \dim_K \text{Soc}_{j-v}(P/J)$ for every $j > v$. The similar results hold for the ideal $\text{in}_\rho(J) \subset P$ and its image I' in $K\{\Gamma\}$. Since $K\{\Gamma\}/I \simeq P/J$ and $K\{\Gamma\}/I' \simeq P/\text{in}_\rho(J)$, it follows that $K\{\Gamma\}/I$ and $K\{\Gamma\}/I'$ have the same Hilbert function, and that $\dim_K(I/mI)_i \leq \dim_k(I'/mI')_i$ and $\dim_K \text{Soc}_i(K\{\Gamma\}/I) \leq \dim_K \text{Soc}_i(K\{\Gamma\}/I')$ for every i.

Thus, replacing I with I' and noting that I' is generated by squarefree monomials, we may assume from the beginning that I itself is generated by squarefree monomials.

Now since I is an ideal in $K\{\Gamma\}$ generated by squarefree monomials, the existence (and uniqueness) of I^{lex} is an immediate consequence of the Kruskal–Katona theorem which, stated in algebraic language, guarantees the following: Suppose that $L \subset K\{\Gamma\}$ is an ideal generated by squarefree monomials, all of the same degree, say q, and let L^{lex} denote the ideal generated by the squarefree lexsegment set $\mathcal{M}$ of degree q with $\sharp(\mathcal{M}) = \dim_K L_q$. Then $\dim_K L_{q+1} \geq \dim_K(L^{\text{lex}})_{q+1}$. Thanks to this fact, given a squarefree ideal I of $K\{\Gamma\}$, if we consider for each i the vector space $V_i \subset K\{\Gamma\}$ spanned by the squarefree lexsegment set $\mathcal{M}_i$ of degree i with $\sharp(\mathcal{M}_i) = \dim_K I_i$, then $\bigoplus_{i \geq 0} V_i$ is an ideal of $K\{\Gamma\}$, which is just the desired I^{lex}. This construction also enables us to see that in each degree the number of generators of I^{lex} is greater than or equal to that of I, which proves the inequalities in (b) for $j = 0$.

Now suppose that $j > 0$. Our hypothesis implies that $m^j I$ and $m^j I^{\text{lex}}$ have the same Hilbert function. Therefore, since $m^j I^{\text{lex}}$ is a lexsegment ideal, we conclude that $(m^j I)^{\text{lex}} = m^j I^{\text{lex}}$. Thus, as above, we deduce from the Kruskal–Katona theorem that in each degree the number of generators of $m^j I^{\text{lex}}$ is greater than or equal to that of $m^j I$. In other words, we have $\dim_K(m^j I/m^{j+1}I) \leq \dim_K(m^j I^{\text{lex}}/m^{j+1}I^{\text{lex}})$. This completes the proof of the inequalities in (b) as desired.

The inequalities (c) will turn out to be again a consequence of the Kruskal–Katona theorem, but not quite as straightforward. Let us first consider the canonical module ω_A of $A = K\{\Gamma\}/I$. We refer the reader to, e.g., [B–H] for basic facts about canonical modules. Since $K\{\Gamma\}$ is a Gorenstein ring (in fact, a complete intersection), we may represent ω_A, up to a shift, as a module of homomorphisms, that is to say, we have $\omega_A(-v) = \text{Hom}_{K\{\Gamma\}}(A, K\{\Gamma\})$. The module $\text{Hom}_{K\{\Gamma\}}(A, K\{\Gamma\})$ of homomorphisms may be naturally identified with the annihilator of I in $K\{\Gamma\}$. Hence, as a graded module, $\omega_A(-v)$ may be regarded as the ideal in $K\{\Gamma\}$ whose K-basis Ω is given by all squarefree monomials $T \in K\{\Gamma\}$ which annihilate I.

We claim that Ω is the set of all squarefree monomials $T^c \in K\{\Gamma\}$ such that $T \notin I$, where T^c is defined by $T^c = x_1 \cdots x_v/T$. In fact, suppose that $T \notin I$ and that $T^c S \neq 0$ for some squarefree monomial $S \in I$. Then, there exists a squarefree monomial U such that $T^c SU = x_1 \cdots x_v = TT^c$. Hence $T = SU$, and thus $T \in I$, a contradiction. Conversely, since $TT^c = x_1 \cdots x_v \neq 0$, if $T^c I = 0$ then $T \notin I$ as desired.

It follows from the identification of $\omega_A(-v)$ with the annihilator of I that $\dim_K(\omega_A)_i = \dim_K A_{-i}$ for every i. Thus, in particular, if $B = K\{\Gamma\}/I^{\text{lex}}$, then the ideals $\omega_A(-v)$ and $\omega_B(-v)$ have the same Hilbert function.

The reader can verify easily, by the above description of the canonical module, that $\omega_B(-v)$ is a squarefree lexsegment ideal of $K\{\Gamma\}$. In fact, if $\mathcal{M}$ is a squarefree lexsegment set of degree q and if $\mathcal{K}$ is the complement of $\mathcal{M}$ in the set of all squarefree monomials of degree q, then the set $\{T^c \,;\, T \in \mathcal{K}\}$ is again a squarefree lexsegment set of degree $v - q$. This observation is crucial, since it implies that $\omega_B(-v) = (\omega_A(-v))^{\text{lex}}$.

Finally we notice that, for every finite dimensional graded K-algebra C and for all integers $i, j \geq 0$, we have $\dim_K(0 :_C m^j)_i = \dim_K(\omega_C/m^j\omega_C)_{-i}$. Thus if we apply the arguments in the proof of (b) to the ideal $\omega_A(-v)$, then the assertion (c) follows. Q. E. D.

By virtue of the Clements–Lindström theorem [C–L], the above Theorem can be generalized to ideals of the quotient algebra $P/(x_1^{a_1}, x_2^{a_2}, \ldots, x_v^{a_v})$ with $1 \leq a_1 \leq a_2 \leq \cdots \leq a_v$.

We now discuss the combinatorial implication of the above result. Let Δ be a simplicial complex on the vertex set $V = \{x_1, x_2, \ldots, x_v\}$, i.e., Δ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq v$ and (ii) if $\sigma \in \Delta$ and $\tau \subset \sigma$ then $\tau \in \Delta$. Each element σ of Δ is called a *face* of Δ. A *facet* of Δ is a face σ of Δ such that $\tau \in \Delta$ and $\sigma \subset \tau$ imply $\sigma = \tau$. Let $f_i = f_i(\Delta)$ be the number of faces σ of Δ with $\sharp(\sigma) = i + 1$, and $n_i = n_i(\Delta)$ the number of facets σ of Δ with $\sharp(\sigma) = i + 1$. Here, $\sharp(\sigma)$ is the cardinality of a finite set σ. Note that $f_{-1} = 1$ and $n_{-1} = 0$. We say that $f(\Delta) = (f_0, f_1, \ldots)$ is the *f-vector* of Δ. Let $P = K[x_1, x_2, \ldots, x_v]$ denote the polynomial ring in v variables over a field K as before, and define I_Δ to be the ideal of P generated by all squarefree monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, with $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} \notin \Delta$. The quotient algebra P/I_Δ is called the *Stanley–Reisner ring* of Δ over K. We refer the reader to, e.g., [B–H], [H], [Hoc] and [Sta] for detailed information about Stanley–Reisner rings.

Now, we write I'_Δ for the image of I_Δ in $K\{\Gamma\}$ and set $K\{\Delta\} = K\{\Gamma\}/I'_\Delta$. Then, the Hilbert function of $K\{\Delta\}$ coincide with the f-vector of Δ, i.e., $\dim_K(K\{\Delta\})_i = f_{i-1}(\Delta)$ for each $i \geq 0$. A simplicial complex Δ is called *lexsegment* if the ideal I'_Δ of $K\{\Gamma\}$ is a squarefree lexsegment ideal. Let Δ^{lex} denote the unique lexsegment simplicial complex with the same f-vector as Δ. Then, $(I'_\Delta)^{\text{lex}} = I'_{\Delta^{\text{lex}}}$.

Recall from, e.g., [H] or [B–H] that, given positive integers f and i, there exists a unique representation of f of the form

$$f = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j},$$
$$a_i > a_{i-1} > \cdots > a_j \geq j \geq 1.$$

We then define

$$\partial_{i-1}(f) = \binom{a_i}{i-1} + \binom{a_{i-1}}{i-2} + \cdots + \binom{a_j}{j-1}.$$

Also, we set $\partial_{i-1}(0) = 0$ for every $i \geq 1$. It is an excercise in combinatorics (see, e.g., [G–K]) to show that, if Δ is a lexsegment simplicial complex with $f(\Delta) = (f_0, f_1, \ldots)$, then $\partial_i(f_i) \leq f_{i-1}$ and $n_{i-1} = f_{i-1} - \partial_i(f_i)$ for each $i \geq 1$.

Corollary. *Let Δ be a simplicial complex with f-vector $(f_0, f_1, \ldots)$. Then, for each $i \geq 1$, we have the inequality $n_{i-1} \leq f_{i-1} - \partial_i(f_i)$. Moreover, when Δ is lexsegment, the equality $n_{i-1} = f_{i-1} - \partial_i(f_i)$ holds for every $i \geq 1$.*

Proof. Since $\dim_K(0 :_{K\{\Delta\}} m)_i = n_{i-1}(\Delta)$, the inequality with $j = 0$ in (c) of the theorem is equal to the required inequality $n_{i-1}(\Delta) \leq n_{i-1}(\Delta^{\text{lex}})$ for every $i \geq 0$. Q. E. D.

The above inequalities are a part of a more general conjecture in [A–H–H] on graded Betti numbers of ideals of the form $(I_\Delta, x_1^2, x_2^2, \ldots, x_v^2)$.

It would, of course, be of interest to find all possible sequences $(n_0, n_1, \ldots)$ arising from the simplicial complexes with a given f-vector.

We now introduce the concept of j-facets of simplicial complexes. We say that a face σ of a simplicial complex Δ is a *j-facet* if j is equal to the greatest integer $k \geq 0$ for which there exists a face τ of Δ such that $\sigma \cap \tau = \emptyset$, $\sigma \cup \tau \in \Delta$ and $\sharp(\tau) = k$. Thus, in particular, the 0-facets of Δ are just the facets of Δ. Let $n_i^j = n_i^j(\Delta)$ denote the number of j-facets σ of Δ with $\sharp(\sigma) = i + 1$. For example, if Δ is a simplicial complex with $f(\Delta) = (5, 7, 1)$, then $n_i^j(\Delta) = n_i^j(\Delta^{\text{lex}})$ for every i and j. On the other hand, let Δ be a simplicial complex on the vertex set $\{x_1, x_2, x_3, x_4\}$ with the facets $\{x_1, x_2\}$ and $\{x_3, x_4\}$. Then, the facets of Δ^{lex} are $\{x_1\}$, $\{x_2, x_4\}$ and $\{x_3, x_4\}$. Hence, $n_0^1(\Delta) = 4$, while $n_0^1(\Delta^{\text{lex}}) = 3$. Hence, in general, the inequality $n_i^j(\Delta) \leq n_i^j(\Delta^{\text{lex}})$ cannot be true if $j \geq 1$. However, we have the following

Corollary. *Let Δ be a simplicial complex and Δ^{lex} the lexsegment simplicial complex with the same f-vector as Δ. Fix $j \geq 0$ and suppose that*

$$n_i^0(\Delta) + n_i^1(\Delta) + \cdots + n_i^{j-1}(\Delta) = n_i^0(\Delta^{\text{lex}}) + n_i^1(\Delta^{\text{lex}}) + \cdots + n_i^{j-1}(\Delta^{\text{lex}})$$

for every $i \geq 0$. Then, we have the inequality

$$n_i^j(\Delta) \leq n_i^j(\Delta^{\text{lex}})$$

for every $i \geq 0$.

Proof. Let $A = K\{\Gamma\}/I'_\Delta$ and $B = K\{\Gamma\}/I'_{\Delta^{\rm lex}} = K\{\Gamma\}/(I'_\Delta)^{\rm lex}$. Then

$$\dim_K(0 :_A m^j)_i = \sum_{k=0}^{j-1} n^k_{i-1}(\Delta)$$

and

$$\dim_K(0 :_B m^j)_i = \sum_{k=0}^{j-1} n^k_{i-1}(\Delta^{\rm lex}).$$

Hence, we can apply (c) of the theorem. Q. E. D.

Let Δ be a lexsegment simplicial complex with $f(\Delta) = (f_0, f_1, \dots)$ and Δ' the subcomplex of Δ obtained by removing all facets of Δ. Then, Δ' is again lexsegment with $f(\Delta') = (\partial_1(f_1), \partial_2(f_2), \dots)$, and, moreover, the facets of Δ' are just the 1-facets of Δ. Hence, it follows that $n^1_{i-1}(\Delta) = \partial_i(f_i) - \partial_i(\partial_{i+1}(f_{i+1}))$ for every i. In general, for every j and i, we have the formula

$$n^j_{i-1}(\Delta) = \partial_i(\partial_{i+1}(\cdots(\partial_{i+j-1}(f_{i+j-1}))\cdots)) - \partial_i(\partial_{i+1}(\cdots(\partial_{i+j}(f_{i+j}))\cdots)).$$

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