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J-HOLOMORPHIC CURVES IN ALMOST COMPLEX SURFACES DO NOT ALWAYS MINIMIZE THE GENUS

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Abstract. The adjunction formula computes the genus of an almost complex curve F embedded in an almost complex surface M in terms of the homology class of F. If M is Kähler (or at least symplectic) and the self-intersection of F is non-negative then the genus of any other surface embedded in M and homologous to F is not less then the genus of F (the proof of this statement (which is a generalization of the Thom conjecture for $\mathbb{C}P^2$) was recently given by the Seiberg-Witten theory). This paper shows that the extra assumptions on M are essential for the genus-minimizing properties of embedded almost complex curves.

Let M be a connected smooth 4-manifold with the tangent bundle τM equipped with a fiberwise-linear map $J : \tau M \to \tau M$ respecting the fibers and such that $J^2 = -1$. In this case M is called an *almost complex surface*. The $\mathbb{R}$-linear map J makes τM into a 2-dimensional complex bundle and induces an orientation on M. The *canonical class* $K \in H^2(M; \mathbb{Z})$ is the Euler class of the exterior square over $\mathbb{C}$ of τM multiplied by (-1).

An embedded surface $F \subset M$ is called a *J-holomorphic curve* if its tangent bundle τF is invariant under J. A J-holomorphic curve gets an orientation from J. If F is J-holomorphic then the normal bundle νF can be chosen to be invariant under J, and the direct sum formula for the characteristic classes of bundles produces the *adjunction formula* for the genus $g(F)$ of F:

$$g(F) = 1 + \frac{F.F + K.F}{2},$$

where $F.F$ denotes the self-intersection of F and $K.F$ denotes the result of evaluation of K on F.

Let $E \subset M$ be an orientable surface homologous to F. The genus $g(E)$ of E is not determined by its homology class. However, in the case when M is symplectic with the symplectic form ω compatible to J so that $\omega(x, Jx) \geq 0$ for any $x \in \tau M$ and $F.F$ is non-negative, the adjunction formula turns into the adjunction inequality

$$g(E) \geq g(F) = 1 + \frac{F.F + K.F}{2}$$

recently proven by means of the Seiberg-Witten theory.

The following theorem shows that the adjunction inequality $g(E) \geq g(F)$ does not hold for all almost complex surfaces.

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Theorem 1. *There exist an almost complex surface M and two smooth closed surfaces $E, F \subset M$, $[E] = [F] \neq 0 \in H_2(M)$, such that F is J-holomorphic and $g(E) < g(F)$.*

Proof. Let M be diffeomorphic to $\mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2$. This manifold admits an almost complex structure with the canonical class $-(3,3,1) \in \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} = H^2(M)$ by Satz 4.6 of [1], since $(3,3,1).(3,3,1) = 19 = 2\chi(M) + 3\sigma(M)$. The next two lemmata produce the required E and F. $\square$

Lemma 1. *There exists a deformation of the almost complex structure on M such that $(4,0,0) \in H_2(M)$ is realizable by a closed J-holomorphic curve F of genus 3.*

Proof. Let $F \subset M$ be the orientable surface of genus 3 produced by an embedding of a nonsingular quartic curve into the first summand $\mathbb{C}P^2$ of $M \approx \mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2$. We may assume (after a small deformation of F) that F is J-holomorphic in a neighbourhood of a point $x \in F$.

The restriction $TM|_F$ of the tangent bundle of M is a trivial 4-dimensional bundle (since $w_1(TM|_F) = 0$ and $w_2(TM|_F) = 0$). Choose a trivialization of $TM|_F$. The subbundle $TF \subset TM|_F$ is then given by a map $\alpha : F \to G_{4,2}$, where $G_{4,2} \approx S^2 \times S^2$ is the Grassmannian of the orientable 2-subspaces in $\mathbb{R}^4$. The homotopy class of such a map is determined by two numbers corresponding to the Euler number of the bundle and the Euler number of the normal bundle.

The J-invariant planes passing through a section of $TF|_{F-\{x\}}$ determine a complex subbundle $B \subset TM|_F$ with the first Chern number $c_1(B) = \chi(F) = -4$. The first Chern number of the bundle normal to B is $16 = F.F$, since $c_1(TM|_F) = (3,3,1).(4,0,0) = 12$. Therefore, the map $\beta : F \to G_{4,2}$ corresponding to B is homotopic to α.

To finish the proof we deform the almost complex structure in the tubular neighbourhood of F by following the homotopy between α and β to make TF J-invariant. $\square$

Lemma 2. *There exists an orientable surface $E \subset M$ of genus 1 realizing the homology class $(4,0,0) \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2)$.*

Remark. By the Rokhlin-Hsiang-Szczarba inequality [5], [2] the genus of any embedded surface of homology class $(4,0,0)$ is at least 1.

Proof of Lemma 2. A nonsingular quartic curve $C \subset \mathbb{C}P^2$ can be obtained from a union of 4 lines and, therefore, it admits a (-1)-membrane for each of its 3 handles (i.e. a disk $M \subset \mathbb{C}P^2$ normal to C along the curve $M \cap C = \partial M$ coinciding with the cocore of the handle and such that the self-intersection number of M equipped with the framing on ∂M coming from C is -1). Making a connected sum with the pair $(\mathbb{C}P^2, \mathbb{C}P^1)$ (cf. [4]) allows us to make the self-intersection of the membrane into zero and, thus, make an embedding surgery removing the handle along this membrane. After repeating this procedure two times we get the required surface $E \subset \mathbb{C}P^2\#\mathbb{C}P^2\#\mathbb{C}P^2 = M$ of genus 1 (cf. getting a $(3,0)$-sphere in $\mathbb{C}P^2\#\mathbb{C}P^2$ in [3]). $\square$

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Remark. As I was informed by the referee, an example similar to the one presented in this paper was given by D. Kotschick in a lecture in Cambridge in 1994.

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