

## THE BEURLING-MALLIAVIN DENSITY OF A RANDOM SEQUENCE

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ABSTRACT. A formula is given for the completeness radius of a random exponential system  $\{t^l e^{i\xi_n t}\}_{l=0, n \in \mathbb{Z}}^{p_n-1}$  in terms of the probability measures of  $\xi_n$ .

The purpose of this note is to formulate a probabilistic version of the celebrated Beurling-Malliavin theorem on the completeness radius of a sequence of complex exponentials [1]. First, to state the Beurling-Malliavin theorem, let  $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$  be a sequence of real numbers and  $\{p_n\}_n$  an associated sequence of positive integers, so that  $p_n$  denotes the multiplicity of  $\lambda_n$  in  $\Lambda$ . The *completeness radius*  $R(\Lambda)$  of  $\mathcal{E}(\Lambda) = \{t^l e^{i\lambda_n t}\}_{l=0, n \in \mathbb{Z}}^{p_n-1}$  is defined as the supremum over all  $R \geq 0$  such that  $\mathcal{E}(\Lambda)$  is a complete sequence in  $L^2(-R, R)$ . The Beurling-Malliavin theorem states that  $R(\Lambda) = \pi D(\Lambda)$ , where  $D(\Lambda)$  is the *Beurling-Malliavin density* of  $\Lambda$ . Following [3], we define this density as follows. A system of intervals  $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ ,  $a_n < b_n$ , is *substantial* if either  $0 < a_1 < b_1 \leq a_2 < b_2 \dots$  and  $\sum_{n \in \mathbb{N}} a_n^{-2} (b_n - a_n)^2 = \infty$ , or  $0 > b_1 > a_1 \geq b_2 > a_2 \dots$  and  $\sum_{n \in \mathbb{N}} |b_n|^{-2} (b_n - a_n)^2 = \infty$ . If  $\Lambda$  has a finite accumulation point, then  $D(\Lambda) = \infty$ . Otherwise,  $D(\Lambda)$  is defined as the supremum over all  $R \geq 0$  for which there exists a substantial sequence of intervals  $\{(a_n, b_n)\}_{n \in \mathbb{N}}$  such that the number of elements of  $\Lambda$  on each interval  $(a_n, b_n)$  is greater than or equal to  $R(b_n - a_n)$ .

If  $\Xi = \{\xi_n\}_{n \in \mathbb{Z}}$  is a sequence of independent random variables, we may study completeness problems for the random exponential system  $\{t^l \exp(i\xi_n t)\}_{l=0, n \in \mathbb{Z}}^{p_n-1}$ , where, as above,  $p_n < \infty$  denotes the multiplicity of the point  $\xi_n$ . Recently, this approach has proved to be fruitful and has led to new insight into classical problems [2], [4]. While [2], [4] deal with random perturbations of a fixed sequence, we now address the problem of computing the Beurling-Malliavin density of an arbitrary random sequence.

To solve this problem, we need the Beurling-Malliavin density of a Borel (not necessarily locally finite) measure. If  $\mu$  is not locally finite, we set  $d(\mu) = \infty$ . Otherwise, the *Beurling-Malliavin density*  $d(\mu)$  is defined as the supremum over all  $R \geq 0$  for which there exists a substantial sequence  $\{(a_n, b_n)\}_{n \in \mathbb{N}}$  such that  $\mu((a_n, b_n)) \geq R(b_n - a_n)$  for every  $n \in \mathbb{N}$ . Observe that to any sequence  $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$  one can associate a measure  $\nu = \sum_{n \in \mathbb{Z}} \delta_{\lambda_n}$ , where  $\delta_x$  is a unit measure concentrated at

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$x$ . We have  $D(\Lambda) = d(\nu)$ , in which sense the density  $d$  is a generalization of the density  $D$ .

We shall prove:

**Theorem.** *Let  $\Xi = \{\xi_n\}_{n \in \mathbb{Z}}$  be a sequence of independent random variables and  $\nu_\Xi = \sum_{n \in \mathbb{Z}} \mu_n$ , where  $\mu_n((-\infty, x)) = \mathbf{P}(\xi_n < x)$  are the probability measures of  $\xi_n$ . Then, with probability one,  $D(\Xi) = d(\nu_\Xi)$ , i.e.,  $R(\Xi) = \pi d(\nu_\Xi)$ .*

Let us denote by  $n_\Lambda(I)$  the number of elements from a sequence  $\Lambda$  in an interval  $I$ . For the proof of the Theorem, we shall need two lemmas.

**Lemma 1.** *Suppose  $\Xi = \{\xi_n\}_{n \in \mathbb{Z}}$  is a sequence of independent random variables such that the measure  $\nu_\Xi$  defined in the Theorem is locally finite. Then the inequality*

$$(1) \quad \mathbf{P}(|n_\Xi(I) - \nu_\Xi(I)| > \alpha) \leq \frac{(2n)!e \max\{1, (\nu_\Xi(I))^n\}}{\alpha^{2n}}$$

holds for all  $\alpha > 0$  and  $n \in \mathbb{N}$ .

*Proof.* Denote by  $\chi_I(x)$  the characteristic function of  $I$ . Then  $n_\Xi(I) = \sum_{n \in \mathbb{Z}} \chi_I(\xi_n)$ . Observe that

$$\mathbf{E}e^{t\chi_I(\xi_n)} = \mu_n(\mathbb{R} \setminus I) + e^t \mu_n(I),$$

where  $\mu_n((-\infty, x)) = \mathbf{P}(\xi_n < x)$  and  $\mathbf{E}$  denotes mathematical expectation. Since the  $\xi_n$  are independent, we obtain

$$\begin{aligned} \mathbf{E}e^{t(n_\Xi(I) - \nu_\Xi(I))} &= e^{-t\nu_\Xi(I)} \prod_{n \in \mathbb{Z}} \mathbf{E}e^{t\chi_I(\xi_n)} = e^{-t\nu_\Xi(I)} \prod_{n \in \mathbb{Z}} (1 + (e^t - 1)\mu_n(I)) \\ &\leq e^{-t\nu_\Xi(I)} \prod_{n \in \mathbb{Z}} e^{(e^t - 1)\mu_n(I)} = e^{(e^t - t - 1)\nu_\Xi(I)}. \end{aligned}$$

Thus, since

$$\frac{2t^{2n}}{(2n)!} (n_\Xi(I) - \nu_\Xi(I))^{2n} < e^{t(n_\Xi(I) - \nu_\Xi(I))} + e^{-t(n_\Xi(I) - \nu_\Xi(I))},$$

we have

$$\begin{aligned} \mathbf{E} (n_\Xi(I) - \nu_\Xi(I))^{2n} &\leq \frac{(2n)!}{2t^{2n}} \mathbf{E}(\exp t(n_\Xi(I) - \nu_\Xi(I)) + \exp \{-t(n_\Xi(I) - \nu_\Xi(I))\}) \\ &\leq \frac{(2n)!}{2t^{2n}} (e^{(e^t - t - 1)\nu_\Xi(I)} + e^{(e^{-t} + t - 1)\nu_\Xi(I)}). \end{aligned}$$

We put  $t = \min\{1, (\nu_\Xi(I))^{-1/2}\}$  and observe that  $e^t - t - 1 < t^2$  for all  $|t| \leq 1$ . Hence

$$\mathbf{E} (n_\Xi(I) - \nu_\Xi(I))^{2n} \leq (2n)!e \max\{1, (\nu_\Xi(I))^n\}.$$

Now (1) follows from Chebyshev's inequality

$$\mathbf{P}(|\eta| > \alpha) < \frac{\mathbf{E}\eta^{2n}}{\alpha^{2n}},$$

which holds for every random variable  $\eta$  with  $\alpha > 0$  and  $n \in \mathbb{N}$ . □

**Lemma 2.** *Let  $\{(a_n, b_n)\}_{n \in \mathbb{N}}$  be a substantial sequence of intervals. For any  $N > 1$  and  $\epsilon > 0$  there exists a set  $S \subset \mathbb{N}$  such that the sequence  $\{(a_n, b_n)\}_{n \in S}$  is substantial,  $b_n - a_n > N$  for all  $n \in S$  and*

$$\sum_{n \in S} (b_n - a_n)^{-3} < \epsilon.$$

*Proof.* Without loss of generality, we assume that our sequence of intervals is such that  $0 < a_1 < b_1 \leq a_2 \dots$  and  $\sum_{n \in \mathbb{N}} a_n^{-2} (b_n - a_n)^2 = \infty$ . Denote by  $Q_N$  the set of all  $n \in \mathbb{N}$  such that  $b_n - a_n \leq N$ . We have

$$\begin{aligned} \sum_{n \in Q_N} \left( \frac{b_n - a_n}{a_n} \right)^2 &\leq N \frac{a_1 + N}{a_1} \sum_{n \in Q_N} \frac{b_n - a_n}{a_n b_n} = N \frac{a_1 + N}{a_1} \sum_{n \in Q_N} \left( \frac{1}{a_n} - \frac{1}{b_n} \right) \\ &\leq N \frac{a_1 + N}{a_1^2} < \infty. \end{aligned}$$

It follows that the sequence  $\{(a_n, b_n)\}_{n \in \mathbb{N} \setminus Q_N}$  is substantial. Denote by  $Q$  the set of  $n \in \mathbb{N}$  such that  $b_n \leq a_n + a_n n^{-5/8}$ . Since

$$\sum_{n \in \mathbb{N}} \left( \frac{b_n - a_n}{a_n} \right)^2 \leq \sum_{n \in \mathbb{N}} n^{-5/4} < \infty,$$

we conclude that the sequence  $\{(a_n, b_n)\}_{n \in \mathbb{N} \setminus (Q \cup Q_N)}$  is substantial. Thus we may assume that  $b_n - a_n > N$  and  $b_n > a_n + a_n n^{-5/8}$  for all  $n \in \mathbb{N}$ . The first of these inequalities yields  $a_n > N(n - 1)$ , and so

$$\sum_{n \in \mathbb{N}} (b_n - a_n)^{-3} \leq (b_1 - a_1)^{-3} + \sum_{n \geq 2} (N(n - 1)n^{-5/8})^{-3} < \infty.$$

Hence, for  $n_\epsilon$  large enough we have  $\sum_{n > n_\epsilon} (b_n - a_n)^{-3} < \epsilon$ , which completes the proof of the lemma.  $\square$

We are now in position to prove the Theorem. Let us denote by  $\Omega$  a probability space on which all the  $\xi_n$  are defined.

(i) Suppose first that the measure  $\nu_\Xi$  is not locally finite, i.e.  $\nu_\Xi(I) = \infty$  for some interval  $I$ . It follows that  $\mathbf{E}n_\Xi(I) = \infty$ . By the Borel-Cantelli Lemma, with probability one the number of  $\xi_n$  which belong to  $I$  is  $\infty$ , so we conclude that with probability one  $R(\Xi) = \infty$ .

(ii) Suppose that  $\nu_\Xi$  is locally finite and  $d(\nu_\Xi) > 0$ . Let us prove that

$$(2) \quad \mathbf{P}(R(\Xi) < d(\nu_\Xi)) = 0.$$

Fix an arbitrary  $\rho$ ,  $0 < \rho < \frac{1}{2}d(\nu_\Xi)$ , and take a substantial sequence of intervals  $\{(a_n, b_n)\}_{n \in \mathbb{Z}}$  such that

$$(b_n - a_n)^{-1} \nu_\Xi((a_n, b_n)) \geq d(\nu_\Xi) - \rho$$

for all  $n$ . By Lemma 2, we may assume that

$$\sum_{n \in \mathbb{N}} (b_n - a_n)^{-3} < \frac{\rho^7}{6!e} (d(\nu_\Xi) - \rho)^3$$

and  $\nu_{\Xi}((a_n, b_n)) > 1$  for all  $n$ . It follows from Lemma 1 that

$$\begin{aligned} & \mathbf{P} (n_{\Xi}(a_n, b_n) \geq (1 - \rho)\nu_{\Xi}((a_n, b_n)) \text{ for all } n) \\ & \geq \mathbf{P} (|n_{\Xi}(a_n, b_n) - \nu_{\Xi}((a_n, b_n))| \leq \rho\nu_{\Xi}((a_n, b_n)) \text{ for all } n) \\ & \geq 1 - \sum_{n \in \mathbb{N}} \mathbf{P} (|n_{\Xi}(a_n, b_n) - \nu_{\Xi}((a_n, b_n))| > \rho\nu_{\Xi}((a_n, b_n))) \\ & > 1 - 6!e\rho^{-6}(d(\nu_{\Xi}) - \rho)^{-3} \sum_{n \in \mathbb{N}} (b_n - a_n)^{-3} > 1 - \rho. \end{aligned}$$

Hence, with probability at least  $1 - \rho$ , we have  $(\nu_{\Xi}((a_n, b_n)))^{-1}n_{\Xi}(a_n, b_n) \geq 1 - \rho$  for all  $n$ . Thus, with probability at least  $1 - \rho$ , we obtain  $R(\Xi) = \infty$  if  $d(\nu_{\Xi}) = \infty$  and  $R(\Xi) \geq (1 - \rho)(d(\nu_{\Xi}) - \rho)$  if  $d(\nu_{\Xi}) < \infty$ . Letting  $\rho \rightarrow 0$ , we obtain (2).

(iii) It remains to be shown that if  $\nu_{\Xi}$  is locally finite and  $d(\nu_{\Xi}) < \infty$ , then

$$(3) \quad \mathbf{P} (R(\Xi) > d(\nu_{\Xi})) = 0.$$

Clearly, we may assume, without loss of generality, that  $\nu_{\Xi}$  is supported on  $\mathbb{R}^+$ .

By Lemma 2, it is enough to consider substantial sequences for which  $a_n > n$ . Since then

$$\sum_{b_n \leq a_n + a_n^{1/4}} \left( \frac{b_n - a_n}{a_n} \right)^2 < \sum_{n \geq 1} n^{-3/2} < \infty,$$

the intervals for which  $b_n > a_n + a_n^{1/4}$  constitute a substantial sequence. By the definition of  $d(\nu_{\Xi})$ , the inequality  $\nu_{\Xi}((a_n, b_n)) \leq (3/2)d(\nu_{\Xi})(b_n - a_n)$  holds for *infinitely many* such  $n$ . Thus, if for some  $\omega \in \Omega$  and  $\epsilon > 0$ , we have

$$R(\Xi(\omega)) \geq d(\nu_{\Xi}) + 2\epsilon,$$

then the inequalities

$$n_{\Xi(\omega)}(k, k + m) > \nu_{\Xi}((k, k + m)) + \epsilon m, \quad \nu_{\Xi}((k, k + m)) \leq 2d(\nu_{\Xi})m$$

hold for infinitely many  $k = 2, 3, \dots$  and some  $m > k^{1/4}$ . It follows that for  $\epsilon > 0$

$$\begin{aligned} & \{\omega : R(\Xi) \geq d(\nu_{\Xi}) + 2\epsilon\} \subset \\ & \bigcap_{j > 1} \bigcup_{k > j} \bigcup_{m > k^{1/4}} \{\omega : |n_{\Xi}(k, k + m) - \nu_{\Xi}((k, k + m))| > \epsilon m, \\ & \nu_{\Xi}((k, k + m)) \leq 2d(\nu_{\Xi})m\}. \end{aligned}$$

Thus, choosing  $n = 6$  in (1), we obtain

$$\mathbf{P}(R(\Xi) \geq d(\nu_{\Xi}) + 2\epsilon) \leq \lim_{j \rightarrow \infty} C\epsilon^{-12}j^{-1/4} = 0.$$

This holds for every  $\epsilon > 0$ , and we have thus proved (3).

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