PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 6, June 1997, Pages 1745–1749
S 0002-9939(97)03750-7

THE BEURLING-MALLIAVIN DENSITY OF A RANDOM SEQUENCE

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(Communicated by J. Marshall Ash)

ABSTRACT. A formula is given for the completeness radius of a random exponential system $\{t^l e^{i\xi_n t}\}_{l=0,n\in\mathbb{Z}}^{p_n-1}$ in terms of the probability measures of ξ_n.

The purpose of this note is to formulate a probabilistic version of the celebrated Beurling-Malliavin theorem on the completeness radius of a sequence of complex exponentials [1]. First, to state the Beurling-Malliavin theorem, let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers and $\{p_n\}_n$ an associated sequence of positive integers, so that p_n denotes the multiplicity of λ_n in Λ. The *completeness radius* $R(\Lambda)$ of $\mathcal{E}(\Lambda) = \{t^l e^{i\lambda_n t}\}_{l=0,n\in\mathbb{Z}}^{p_n-1}$ is defined as the supremum over all $R \geq 0$ such that $\mathcal{E}(\Lambda)$ is a complete sequence in $L^2(-R,R)$. The Beurling-Malliavin theorem states that $R(\Lambda) = \pi D(\Lambda)$, where $D(\Lambda)$ is the *Beurling-Malliavin density* of Λ. Following [3], we define this density as follows. A system of intervals $\{(a_n,b_n)\}_{n\in\mathbb{N}}$, $a_n < b_n$, is *substantial* if either $0 < a_1 < b_1 \leq a_2 < b_2 \ldots$ and $\sum_{n\in\mathbb{N}} a_n^{-2}(b_n - a_n)^2 = \infty$, or $0 > b_1 > a_1 \geq b_2 > a_2 \ldots$ and $\sum_{n\in\mathbb{N}} |b_n|^{-2}(b_n-a_n)^2 = \infty$. If Λ has a finite accumulation point, then $D(\Lambda) = \infty$. Otherwise, $D(\Lambda)$ is defined as the supremum over all $R \geq 0$ for which there exists a substantial sequence of intervals $\{(a_n,b_n)\}_{n\in\mathbb{N}}$ such that the number of elements of Λ on each interval (a_n,b_n) is greater than or equal to $R(b_n - a_n)$.

If $\Xi = \{\xi_n\}_{n\in\mathbb{Z}}$ is a sequence of independent random variables, we may study completeness problems for the random exponential system $\{t^l \exp(i\xi_n t)\}_{l=0,n\in\mathbb{Z}}^{p_n-1}$, where, as above, $p_n < \infty$ denotes the multiplicity of the point ξ_n. Recently, this approach has proved to be fruitful and has led to new insight into classical problems [2], [4]. While [2], [4] deal with random perturbations of a fixed sequence, we now address the problem of computing the Beurling-Malliavin density of an arbitrary random sequence.

To solve this problem, we need the Beurling-Malliavin density of a Borel (not necessarily locally finite) measure. If μ is not locally finite, we set $d(\mu) = \infty$. Otherwise, the *Beurling-Malliavin density* $d(\mu)$ is defined as the supremum over all $R \geq 0$ for which there exists a substantial sequence $\{(a_n,b_n)\}_{n\in\mathbb{N}}$ such that $\mu((a_n,b_n)) \geq R(b_n - a_n)$ for every $n \in \mathbb{N}$. Observe that to any sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ one can associate a measure $\nu = \sum_{n\in\mathbb{Z}} \delta_{\lambda_n}$, where δ_x is a unit measure concentrated at

Received by the editors September 8, 1995 and, in revised form, December 11, 1995.
1991 *Mathematics Subject Classification.* Primary 42A61, 42C30.
Research supported by NATO linkage grant LG 930329.

x. We have $D(\Lambda) = d(\nu)$, in which sense the density d is a generalization of the density D.

We shall prove:

Theorem. *Let $\Xi = \{\xi_n\}_{n\in\mathbb{Z}}$ be a sequence of independent random variables and $\nu_\Xi = \sum_{n\in\mathbb{Z}} \mu_n$, where $\mu_n((-\infty, x)) = \mathbf{P}(\xi_n < x)$ are the probability measures of ξ_n. Then, with probability one, $D(\Xi) = d(\nu_\Xi)$, i.e., $R(\Xi) = \pi d(\nu_\Xi)$.*

Let us denote by $n_\Lambda(I)$ the number of elements from a sequence Λ in an interval I. For the proof of the Theorem, we shall need two lemmas.

Lemma 1. *Suppose $\Xi = \{\xi_n\}_{n\in\mathbb{Z}}$ is a sequence of independent random variables such that the measure ν_Ξ defined in the Theorem is locally finite. Then the inequality*

$$\mathbf{P}\left(|n_\Xi(I) - \nu_\Xi(I)| > \alpha\right) \le \frac{(2n)!e\max\{1, (\nu_\Xi(I))^n\}}{\alpha^{2n}} \tag{1}$$

holds for all $\alpha > 0$ and $n \in \mathbb{N}$.

Proof. Denote by $\chi_I(x)$ the characteristic function of I. Then $n_\Xi(I) = \sum_{n\in\mathbb{Z}} \chi_I(\xi_n)$. Observe that

$$\mathbf{E}e^{t\chi_I(\xi_n)} = \mu_n(\mathbb{R}\setminus I) + e^t\mu_n(I),$$

where $\mu_n((-\infty, x)) = \mathbf{P}(\xi_n < x)$ and $\mathbf{E}$ denotes mathematical expectation. Since the ξ_n are independent, we obtain

$$\mathbf{E}e^{t(n_\Xi(I)-\nu_\Xi(I))} = e^{-t\nu_\Xi(I)}\prod_{n\in\mathbb{Z}}\mathbf{E}e^{t\chi_I(\xi_n)} = e^{-t\nu_\Xi(I)}\prod_{n\in\mathbb{Z}}\left(1 + (e^t - 1)\mu_n(I)\right)$$

$$\le e^{-t\nu_\Xi(I)}\prod_{n\in\mathbb{Z}} e^{(e^t-1)\mu_n(I)} = e^{(e^t-t-1)\nu_\Xi(I)}.$$

Thus, since

$$\frac{2t^{2n}}{(2n)!}\left(n_\Xi(I) - \nu_\Xi(I)\right)^{2n} < e^{t(n_\Xi(I)-\nu_\Xi(I))} + e^{-t(n_\Xi(I)-\nu_\Xi(I))},$$

we have

$$\mathbf{E}\left(n_\Xi(I) - \nu_\Xi(I)\right)^{2n} \le \frac{(2n)!}{2t^{2n}}\mathbf{E}(\exp t(n_\Xi(I) - \nu_\Xi(I)) + \exp\{-t(n_\Xi(I) - \nu_\Xi(I))\})$$

$$\le \frac{(2n)!}{2t^{2n}}(e^{(e^t-t-1)\nu_\Xi(I)} + e^{(e^{-t}+t-1)\nu_\Xi(I)}).$$

We put $t = \min\{1, (\nu_\Xi(I))^{-1/2}\}$ and observe that $e^t - t - 1 < t^2$ for all $|t| \le 1$. Hence

$$\mathbf{E}\left(n_\Xi(I) - \nu_\Xi(I)\right)^{2n} \le (2n)!e\max\{1, (\nu_\Xi(I))^n\}.$$

Now (1) follows from Chebyshev's inequality

$$\mathbf{P}(|\eta| > \alpha) < \frac{\mathbf{E}\eta^{2n}}{\alpha^{2n}},$$

which holds for every random variable η with $\alpha > 0$ and $n \in \mathbb{N}$. ∎

Lemma 2. *Let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ be a substantial sequence of intervals. For any $N > 1$ and $\epsilon > 0$ there exists a set $S \subset \mathbb{N}$ such that the sequence $\{(a_n, b_n)\}_{n\in S}$ is substantial, $b_n - a_n > N$ for all $n \in S$ and*

$$\sum_{n\in S}(b_n - a_n)^{-3} < \epsilon.$$

Proof. Without loss of generality, we assume that our sequence of intervals is such that $0 < a_1 < b_1 \le a_2 ...$ and $\sum_{n\in\mathbb{N}} a_n^{-2}(b_n - a_n)^2 = \infty$. Denote by Q_N the set of all $n \in \mathbb{N}$ such that $b_n - a_n \le N$. We have

$$\sum_{n\in Q_N}\left(\frac{b_n - a_n}{a_n}\right)^2 \le N\frac{a_1 + N}{a_1}\sum_{n\in Q_N}\frac{b_n - a_n}{a_n b_n} = N\frac{a_1 + N}{a_1}\sum_{n\in Q_N}\left(\frac{1}{a_n} - \frac{1}{b_n}\right)$$

$$\le N\frac{a_1 + N}{a_1^2} < \infty.$$

It follows that the sequence $\{(a_n, b_n)\}_{n\in\mathbb{N}\setminus Q_N}$ is substantial. Denote by Q the set of $n \in \mathbb{N}$ such that $b_n \le a_n + a_n n^{-5/8}$. Since

$$\sum_{n\in\mathbb{N}}\left(\frac{b_n - a_n}{a_n}\right)^2 \le \sum_{n\in\mathbb{N}} n^{-5/4} < \infty,$$

we conclude that the sequence $\{(a_n, b_n)\}_{n\in\mathbb{N}\setminus(Q\bigcup Q_N)}$ is substantial. Thus we may assume that $b_n - a_n > N$ and $b_n > a_n + a_n n^{-5/8}$ for all $n \in \mathbb{N}$. The first of these inequalities yields $a_n > N(n-1)$, and so

$$\sum_{n\in\mathbb{N}}(b_n - a_n)^{-3} \le (b_1 - a_1)^{-3} + \sum_{n\ge 2}(N(n-1)n^{-5/8})^{-3} < \infty.$$

Hence, for n_ϵ large enough we have $\sum_{n>n_\epsilon}(b_n - a_n)^{-3} < \epsilon$, which completes the proof of the lemma. □

We are now in position to prove the Theorem. Let us denote by Ω a probability space on which all the ξ_n are defined.

(i) Suppose first that the measure ν_Ξ is not locally finite, i.e. $\nu_\Xi(I) = \infty$ for some interval I. It follows that $\mathbf{E}n_\Xi(I) = \infty$. By the Borel-Cantelli Lemma, with probability one the number of ξ_n which belong to I is ∞, so we conclude that with probability one $R(\Xi) = \infty$.

(ii) Suppose that ν_Ξ is locally finite and $d(\nu_\Xi) > 0$. Let us prove that

$$\mathbf{P}\left(R(\Xi) < d(\nu_\Xi)\right) = 0. \tag{2}$$

Fix an arbitrary ρ, $0 < \rho < \frac{1}{2}d(\nu_\Xi)$, and take a substantial sequence of intervals $\{(a_n, b_n)\}_{n\in\mathbb{Z}}$ such that

$$(b_n - a_n)^{-1}\nu_\Xi\big((a_n, b_n)\big) \ge d(\nu_\Xi) - \rho$$

for all n. By Lemma 2, we may assume that

$$\sum_{n\in\mathbb{N}}(b_n - a_n)^{-3} < \frac{\rho^7}{6!e}(d(\nu_\Xi) - \rho)^3$$

and $\nu_\Xi((a_n,b_n)) > 1$ for all n. It follows from Lemma 1 that

$$\mathbf{P}\left(n_\Xi(a_n,b_n) \geq (1-\rho)\nu_\Xi\big((a_n,b_n)\big) \text{ for all } n\right)$$

$$\geq \mathbf{P}\left(|n_\Xi(a_n,b_n) - \nu_\Xi\big((a_n,b_n)\big)| \leq \rho\nu_\Xi\big((a_n,b_n)\big) \text{ for all } n\right)$$

$$\geq 1 - \sum_{n\in\mathbb{N}} \mathbf{P}\left(|n_\Xi(a_n,b_n) - \nu_\Xi\big((a_n,b_n)\big)| > \rho\nu_\Xi\big((a_n,b_n)\big)\right)$$

$$> 1 - 6!e\rho^{-6}(d(\nu_\Xi)-\rho)^{-3}\sum_{n\in\mathbb{N}}(b_n-a_n)^{-3} > 1-\rho.$$

Hence, with probability at least $1-\rho$, we have $\big(\nu_\Xi((a_n,b_n))\big)^{-1} n_\Xi(a_n,b_n) \geq 1-\rho$ for all n. Thus, with probability at least $1-\rho$, we obtain $R(\Xi)=\infty$ if $d(\nu_\Xi)=\infty$ and $R(\Xi) \geq (1-\rho)(d(\nu_\Xi)-\rho)$ if $d(\nu_\Xi)<\infty$. Letting $\rho\to 0$, we obtain (2).

(iii) It remains to be shown that if ν_Ξ is locally finite and $d(\nu_\Xi)<\infty$, then

$$\mathbf{P}\left(R(\Xi) > d(\nu_\Xi)\right) = 0. \tag{3}$$

Clearly, we may assume, without loss of generality, that ν_Ξ is supported on $\mathbb{R}^+$.

By Lemma 2, it is enough to consider substantial sequences for which $a_n > n$. Since then

$$\sum_{b_n \leq a_n + a_n^{1/4}} \left(\frac{b_n-a_n}{a_n}\right)^2 < \sum_{n\geq 1} n^{-3/2} < \infty,$$

the intervals for which $b_n > a_n + a_n^{1/4}$ constitute a substantial sequence. By the definition of $d(\nu_\Xi)$, the inequality $\nu_\Xi((a_n,b_n)) \leq (3/2)d(\nu_\Xi)(b_n-a_n)$ holds for *infinitely many* such n. Thus, if for some $\omega\in\Omega$ and $\epsilon>0$, we have

$$R(\Xi(\omega)) \geq d(\nu_\Xi) + 2\epsilon,$$

then the inequalities

$$n_{\Xi(\omega)}(k,k+m) > \nu_\Xi\big((k,k+m)\big) + \epsilon m, \;\; \nu_\Xi\big((k,k+m)\big) \leq 2d(\nu_\Xi)m$$

hold for infinitely many $k = 2,3,\ldots$ and some $m > k^{1/4}$. It follows that for $\epsilon>0$

$$\{\omega:\; R(\Xi) \geq d(\nu_\Xi)+2\epsilon\} \subset \bigcap_{j>1}\bigcup_{k>j}\bigcup_{m>k^{1/4}} \{\omega:\; |n_\Xi(k,k+m) - \nu_\Xi((k,k+m))| > \epsilon m, \;\; \nu_\Xi((k,k+m)) \leq 2d(\nu_\Xi)m\}.$$

Thus, choosing $n=6$ in (1), we obtain

$$\mathbf{P}(R(\Xi) \geq d(\nu_\Xi)+2\epsilon) \leq \lim_{j\to\infty} C\epsilon^{-12}j^{-1/4} = 0.$$

This holds for every $\epsilon>0$, and we have thus proved (3).

Acknowledgement

The authors wish to thank the referee for pertinent comments, which helped to improve the paper.

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