PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 6, June 1997, Pages 1841–1843
S 0002-9939(97)03754-4

AMENABLE REPRESENTATIONS AND FINITE INJECTIVE VON NEUMANN ALGEBRAS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let $U(M)$ be the unitary group of a finite, injective von Neumann algebra M. We observe that any subrepresentation of a group representation into $U(M)$ is amenable in the sense of Bekka; this yields short proofs of two known results—one by Robertson, one by Haagerup—concerning group representations into $U(M)$.

A unitary representation π of a group Γ on a Hilbert space $\mathcal{H}_\pi$ is *amenable* if there exists on $\mathcal{B}(\mathcal{H}_\pi)$ an $\operatorname{Ad}\pi$-invariant state, i.e. a state ϕ on $\mathcal{B}(\mathcal{H}_\pi)$ such that, for any $T \in \mathcal{B}(\mathcal{H}_\pi),\ g \in \Gamma$:

$$\phi(\pi(g)T\pi(g^{-1})) = \phi(T).$$

This notion was introduced and studied by Bekka in [Be].

In the present paper, M will always denote a finite, injective von Neumann algebra. We start with the observation that, if $\pi(\Gamma)$ is contained in the unitary group $U(M)$, then any subrepresentation of π is amenable. We use this to give short, hopefully new proofs of two known results. The first, due to Robertson ([Ro], Theorem C and remark (4) on p. 554), states that for any representation π of a group Γ with Kazhdan's property (T) into $U(M)$, the closure of $\pi(\Gamma)$ in the L^2-norm topology on $U(M)$ is compact. The second, due to Haagerup ([Ha], Lemma 2.2), says that for any $n \in \mathbb{N}$, $U_1, U_2, \ldots, U_n \in U(M)$ and P a non-zero projector in the commutant M' of M:

$$\left\| \sum_{i=1}^{n} PU_i \otimes \bar{P}\bar{U}_i \right\| = n$$

(where the bar denotes the same operator, but acting on the conjugate Hilbert space).

Proposition 1. *Let π be a representation of a group Γ into $U(M)$. Then any subrepresentation of π is amenable.*

Proof. We may assume that $M = \pi(\Gamma)''$. Let ρ be a subrepresentation of π on a closed subspace $\mathcal{H}_\rho$ which is the range of a projector $p \in M'$. To construct an $\operatorname{Ad}\rho$-invariant state on $\mathcal{B}(\mathcal{H}_\rho) = p\mathcal{B}(\mathcal{H}_\pi)p$, choose a conditional expectation

Received by the editors October 6, 1995 and, in revised form, December 5, 1995.
1991 *Mathematics Subject Classification.* Primary 22D25; Secondary 46L10.
Key words and phrases. Amenable representations, finite injective von Neumann algebra, Kazhdan's property (T).

$E\colon \mathcal{B}(\mathcal{H}_\rho) \to pM$, a trace τ on pM, and set $\phi = \tau \circ E$. Then, for any $T \in \mathcal{B}(\mathcal{H}_\rho)$, $g \in \Gamma$:

$$\phi(\rho(g)T\rho(g^{-1})) = \tau(\rho(g)E(T)\rho(g^{-1})) = \phi(T),$$

which concludes the proof. □

For a representation π of Γ into $U(M)$, the closure of $\pi(\Gamma)$ in $U(M)$ is compact in the L^2-norm if and only if π decomposes as a direct sum of finite-dimensional representations. Indeed, if $\pi(\Gamma)$ is relatively compact, the identity representation of the compact group $\overline{\pi(\Gamma)}$ decomposes into finite-dimensional representations; conversely, using the fact that the L^2 and strong topologies coincide on $U(M)$, it is easy to see that any unitary representation that decomposes into finite-dimensional ones, has relatively compact range. We then state Robertson's result mentioned above in the following equivalent form.

Proposition 2. *Let Γ be a group with Kazhdan's property* (T)*. Any representation π of Γ into $U(M)$ decomposes as a direct sum of finite-dimensional representations.*

Proof. By Zorn's lemma, find a subrepresentation ρ of π that decomposes as a direct sum of finite-dimensional representations, and maximal with respect to that property. We have to show that $\rho = \pi$. If this is not the case, consider the subrepresentation $\rho^\perp$ on the orthogonal complement of $\mathcal{H}_\rho$. By Proposition 1, $\rho^\perp$ is an amenable representation of Γ. Because Γ has property (T), it follows from Corollary 5.9 of [Be] that any amenable representation of Γ has a (non-zero) finite-dimensional subrepresentation. This, applied to $\rho^\perp$, contradicts maximality of ρ. □

Remarks. (1) Proposition 2 makes precise an earlier result of Kirchberg ([Ki], Corollary 1.2): any Kazhdan group that admits a faithful representation into $U(M)$, must be residually finite. Kirchberg also mentions in the same paper [Ki] that his proof of residual finiteness for certain subgroups of $U(M)$ works for a bigger class of subgroups than just subgroups with property (T) (e.g. it works for non-abelian free groups). However, the proof of Proposition 2 does not extend, in view of Theorem 1 of [BV]: Kazhdan groups are characterized by the fact that any amenable representation has a finite-dimensional subrepresentation.

(2) Robertson proved ([Ro], Lemma 4.2) a result more general than our Proposition 2: any representation of a Kazhdan group into the unitary group of a finite von Neumann algebra with Haagerup's approximation property, decomposes as a direct sum of finite-dimensional representations.

We now turn to Haagerup's result mentioned in the beginning.

Proposition 3. *For any $n \in \mathbb{N}$, $U_1, U_2, \ldots, U_n \in U(M)$ and P a non-zero projector in the commutant M' of M:*

$$\left\| \sum_{i=1}^{n} PU_i \otimes \bar{P}\bar{U}_i \right\| = n.$$

Proof. Let Γ be a finitely generated group, S a finite generating subset, and π a unitary representation of Γ. It follows from Theorem 5.1 of [Be] and remark (b) on p. 74 of [HRV], that π is amenable if and only if 1 belongs to the spectrum of the operator $\frac{1}{|S|}\sum_{s\in S}(\pi \otimes \bar{\pi})(s)$, where $\bar{\pi}$ denotes the contragredient representation of π. If this is the case, then $\|\sum_{s\in S}(\pi\otimes\bar{\pi})(s)\| = |S|$. The result then follows by considering the U_i's as generators of a representation of the free group $\mathbb{F}_n$ on n generators, and appealing to Proposition 1. □

Remark. Haagerup has proved that Proposition 3 actually yields a characterization of finite, injective von Neumann algebras. Indeed, it is enough to assume $\| \sum_{i=1}^n PU_i \otimes \bar{P}\bar{U}_i\| = n$ for any n-tuple of unitaries and any non-zero *central* projection, to make sure that M is finite and injective (see [Ha], Lemma 2.2). For a factor acting on a separable Hilbert space, this characterization of the hyperfinite II_1-factor is due to Connes ([Co], Remark 5.29).

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