

## FINITE DIMENSIONAL REPRESENTATIONS OF $U_q(sl(2, 1))$

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ABSTRACT. Structures of the finite dimensional simple weight  $U_q(sl(2, 1))$ -modules are studied in detail for both the generic and the roots of 1 cases.

### 1. INTRODUCTION

Over a field of characteristic 0, the representation theory of the quantized enveloping algebra  $U_q(G_0)$  of a semisimple Lie algebra  $G_0$  for a generic  $q$  is remarkably similar to the representation theory of the enveloping algebra  $U(G_0)$ . By a well-known result of Lusztig [8], any simple integrable highest weight  $U(G_0)$ -module admits a quantum deformation, and the characters of the finite dimensional simple highest weight  $U_q(G_0)$ -modules are given by the Weyl-Kac character formula. On the contrary, when  $q$  is a root of 1, the representations of  $U_q(G_0)$  are much more complicated. Just like the representation theory of  $U_q(G_0)$ , the representation theory of the quantized enveloping algebras of Lie superalgebras has many interesting applications such as constructing link invariants (see for example [7] and [12]). Therefore a systematic study of the representation theory of the quantized enveloping algebras of Lie superalgebras is desirable.

The basic idea of [1], [8] and [9] can also be applied to the case of a quantized enveloping algebra  $U_q(G)$  of a contragredient Lie superalgebra  $G$ , especially the classical ones (see [5]). Along this line, one can analyze the highest weight  $U_q(G)$ -modules which are deformations of the corresponding  $U(G)$ -modules. Some results have been obtained by several authors (see [4], [13] and the references therein). However, a full account of the study of the finite dimensional simple modules for these algebras, especially for the roots of 1 case, is still pending even for the algebra  $sl(2, 1)$ .

The aim of this paper is to study the structures of all finite dimensional simple weight modules for  $U_q(sl(2, 1))$  (see our definition of  $U_q(sl(2, 1))$  in section 2) for both the generic case and the roots of 1 case over  $\mathbb{C}$ . We show that any finite dimensional weight module for these algebras, regardless whether  $q$  is a root of 1 or not, is the quotient of some module induced from a simple weight module of the even part (note that a simple weight module for the even part is not necessarily a highest weight module if  $q$  is a root of 1). These induced modules are the  $q$ -analogs of the so-called Kac modules. We then analyze the structures of these induced modules to obtain a complete description of the finite dimensional simple weight modules.

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In particular, for a generic  $q$ , our result shows that every finite dimensional simple highest weight  $U(\mathfrak{sl}(2, 1))$ -module admits a  $q$ -deformation.

We arrange this paper as follows. In section 2, the definition of the version of  $U_q(\mathfrak{sl}(2, 1))$  adopted in this paper will be given, and its structure will be discussed. In section 3, we show that any finite dimensional simple weight module is a quotient of some induced module and give a sufficient condition for these induced modules to be simple. In section 4, we analyze the structures of the induced modules for the generic case; the analysis also provides a clear picture of the structures of finite dimensional simple weight modules in this case. Finally, in section 5, we treat the roots of 1 case.

## 2. ALGEBRAS $U$ AND $U_\varepsilon$

The deformations of some of the classical Lie superalgebras have been discussed by several authors (see [2], [3], [10], and [14]). We first recall the definition of  $U_q(\mathfrak{sl}(2, 1))$ . Let  $q$  be an indeterminate over the complex number field  $\mathbb{C}$ , let  $G = \mathfrak{sl}(2, 1)$ , and let  $(a_{ij})_{2 \times 2}$  be defined by  $a_{11} = 2, a_{22} = 0$  and  $a_{12} = a_{21} = -1$ . Then  $U = U_q(G)$  is the associative  $\mathbb{Z}_2$ -graded algebra over  $\mathbb{C}(q)$  (with 1) generated by  $e_i, f_i, t_i^{\pm 1}, i = 1, 2$ , with the grading given by  $\deg(e_1) = \deg(f_1) = \deg(t_1^{\pm 1}) = \deg(t_2^{\pm 1}) = 0, \deg(e_2) = \deg(f_2) = 1$ , such that the following conditions hold:

- (1)  $t_i t_i^{-1} = t_i^{-1} t_i = 1,$
- (2)  $t_i t_j = t_j t_i,$
- (3)  $t_i e_j t_i^{-1} = q^{a_{ij}} e_j,$
- (4)  $t_i f_j t_i^{-1} = q^{-a_{ij}} f_j,$
- (5)  $e_i f_j - (-1)^{ab} f_j e_i = \delta_{ij} (t_i - t_i^{-1}) / (q - q^{-1}),$  where  $a = \deg(e_i), b = \deg(f_j),$
- (6)  $e_1^2 e_2 - (q + q^{-1}) e_1 e_2 e_1 + e_2 e_1^2 = 0, f_1^2 f_2 - (q + q^{-1}) f_1 f_2 f_1 + f_2 f_1^2 = 0,$
- (7)  $e_2^2 = 0, f_2^2 = 0.$

The algebra  $U$  is a graded Hopf algebra with comultiplication  $\Delta,$  antipode  $S$  and counit  $\epsilon$  defined by

- (8)  $\Delta t_i^{\pm 1} = t_i^{\pm 1} \otimes t_i^{\pm 1}, \Delta e_i = e_i \otimes 1 + t_i \otimes e_i, \Delta f_i = f_i \otimes t_i^{-1} + 1 \otimes f_i,$
- (9)  $S t_i = t_i^{-1}, S e_i = -t_i^{-1} e_i, S f_i = -f_i t_i,$
- (10)  $\epsilon(e_i) = \epsilon(f_i) = 0, \epsilon(t_i^{\pm 1}) = 1,$

where  $i = 1, 2.$  The adjoint action is defined by

$$ad_q = m(m_L \otimes m_R)(id \otimes S)\Delta,$$

where  $m_L$  and  $m_R$  are the left and the right graded multiplications given by

$$m_L(x)y = xy, \quad m_R(x)y = (-1)^{\deg(x)\deg(y)}yx,$$

and  $m : U \otimes U \rightarrow U$  is the usual multiplication.

Define an anti-automorphism  $\omega : U \rightarrow U$  by

$$\omega e_i = f_i, \omega f_i = e_i, \omega t_i = t_i^{-1}, \omega q = q^{-1},$$

and  $\omega(xy) = \omega(y)\omega(x)$  for any  $x, y \in U.$  Note that the  $\omega$  thus defined preserves the generating relations (1)-(7), while the one defined by a twist given by

$$\omega(xy) = (-1)^{\deg(x)\deg(y)}\omega(y)\omega(x)$$

does not preserve condition (5) for  $e_2$  and  $f_2.$

We introduce the root vectors

$$e_3 = qe_1e_2 - e_2e_1, \quad f_3 = \omega(e_3) = -f_1f_2 + q^{-1}f_2f_1.$$

**Lemma 2.1.** *The following identities hold in  $U$ :*

- 1)  $e_3^2 = 0, f_3^2 = 0$ ;
- 2)  $e_1e_3 = qe_3e_1, e_2e_3 = -qe_3e_2$ ;
- 3)  $f_1f_3 = qf_3f_1, f_2f_3 = -qf_3f_2$ .

*Proof.* We verify the identities for the  $e_i$ 's; the identities for the  $f_i$ 's are then obtained by using  $\omega$ . We first verify 2). The identity  $e_2e_3 = -qe_3e_2$  follows from a simple computation. By using (6), we have

$$e_1e_3 - qe_3e_1 = (ad_q e_1)^2(e_2) = 0.$$

Hence 2) follows. To verify 1), use 2); we have

$$\begin{aligned} 0 &= ad_q e_1(e_2e_3 + qe_3e_2) \\ &= e_1e_2e_3 + qe_1e_3e_2 - e_2e_3e_1 - qe_3e_2e_1 \\ &= e_1e_2e_3 - q^{-1}e_2e_1e_3 + q^2e_3e_1e_2 - qe_3e_2e_1 \\ &= q^{-1}e_3^2 + qe_3^2 = (q^{-1} + q)e_3^2, \end{aligned}$$

and hence  $e_3^2 = 0$ . □

Straightforward computations show that the following relations also hold in  $U$ :

- Lemma 2.2.** 1)  $f_1e_3 - e_3f_1 = t_1^{-1}e_2, f_3e_1 - e_1f_3 = f_2t_1$ ;  
 2)  $f_2e_3 + e_3f_2 = qt_2e_1, f_3e_2 + e_2f_3 = t_2^{-1}f_1$ ;  
 3)  $f_3e_3 + e_3f_3 = t_2H_1 + t_1^{-1}H_2 = t_1H_2 + t_2^{-1}H_1$ ,  
 where  $H_i = (t_i - t_i^{-1})/(q - q^{-1}), i = 1, 2$ .

We also have

**Lemma 2.3.** *The following formulas hold in  $U$ :*

$$f_1^k f_2 = q^{-k} f_2 f_1^k - [k] f_3 f_1^{k-1}, \quad e_1^k e_2 = q^{-k} e_2 e_1^k + [k] q^{-1} e_3 e_1^{k-1},$$

where  $[k] = (q^k - q^{-k})/(q - q^{-1})$ .

*Proof.* The first formula follows by using induction on  $k$ ; the second formula can then be obtained by using the anti-automorphism  $\omega$ . □

Let  $U^+, U^-$  and  $U^0$  be the subalgebras (with 1) of  $U$  generated by the  $e_i$ , the  $f_i$  and the  $t_i^{\pm 1}$  ( $i = 1, 2$ ) respectively. Let  $U^{\geq 0} = U^+ U^0$  (it is easy to see that this is a subalgebra of  $U$ ), let  $P$  be generated by  $U^{\geq 0}$  together with  $f_1$ , and let  $U_0$  be generated by  $e_1, f_1, t_1^{\pm 1}$  and  $t_2^{\pm 1}$ . Then elements of the forms  $e_3^{d_3} e_2^{d_2} e_1^k$  (resp.  $f_3^{d_3} f_2^{d_2} f_1^k$ ),  $d_3, d_2 \in \{0, 1\}, k \in \mathbb{Z}_+$ , form a basis of  $U^+$  (resp.  $U^-$ ), and the monomials  $t_2^{m_2} t_1^{m_1}, m_1, m_2 \in \mathbb{Z}$ , form a basis of  $U^0$ . Let  $r = (d_1, d_2, k) \in \{0, 1\} \times \{0, 1\} \times \mathbb{Z}_+, m = (m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$ , and let  $e^r = e_3^{d_1} e_2^{d_2} e_1^k, f^r = f_3^{d_1} f_2^{d_2} f_1^k, t^m = t_1^{m_1} t_2^{m_2}$ . Then PBW theorem says that elements of the form  $f^{r'} t^m e^r$  form a basis of  $U$ .

Let  $\mathcal{A} = \mathbb{C}[q, q^{-1}]$ , and let  $U_{\mathcal{A}}$  be the  $\mathcal{A}$ -subalgebra of  $U$  generated by the elements  $e_i, f_i, t_i^{\pm 1}$  ( $i = 1, 2$ ). Let  $U_{\mathcal{A}}^+$  (resp.  $U_{\mathcal{A}}^-$ ) be the  $\mathcal{A}$ -subalgebra of  $U_{\mathcal{A}}$  generated by the  $e_i$  (resp.  $f_i$ ), and let  $U_{\mathcal{A}}^0$  be generated by the  $t_i$  and the  $H_i$ . The relations (1)-(4), (6), (7) together with

$$(2.1) \quad e_i f_j - f_j e_i = \delta_{ij} H_i,$$

$$(2.2) \quad (q - q^{-1}) H_i = t_i - t_i^{-1},$$

give the defining relations of the algebra  $U_{\mathcal{A}}$ . The algebra  $U_{\mathcal{A}}$  is a graded Hopf algebra with  $\Delta$ ,  $S$  and  $\epsilon$  defined by (8)-(10) and

$$\Delta H_i = H_i \otimes t_i + t_i^{-1} \otimes H_i, \quad SH_i = -H_i, \quad \epsilon H_i = 0.$$

For  $\varepsilon \in \mathbb{C}^*$ , we let  $U_\varepsilon = U_{\mathcal{A}}/(q - \varepsilon)U_{\mathcal{A}}$ . We identify  $e_i, f_i, t_i$  ( $i = 1, 2$ ), with their images in  $U_\varepsilon$ , and denote by  $U_\varepsilon^+, U_\varepsilon^-, U_\varepsilon^0$  the images of  $U_{\mathcal{A}}^+, U_{\mathcal{A}}^-$  and  $U_{\mathcal{A}}^0$  in  $U_\varepsilon$  respectively. We also let  $U_\varepsilon^{\geq 0} = U_\varepsilon^0 U_\varepsilon^+$ , let  $P_\varepsilon$  be the subalgebra of  $U_\varepsilon$  generated by  $U_\varepsilon^{\geq 0}$  together with  $f_1$ , and let  $U_{\varepsilon,0}$  be generated by  $e_1, f_1$  and  $U_\varepsilon^0$ . Note that we have  $U_1/(t_i - 1; i = 1, 2) \cong U(G)$ .

### 3. FINITE DIMENSIONAL INDUCED WEIGHT MODULES

We first recall some facts about  $G$  and its representations.

Let  $G = N^- + H + N^+$  be the standard triangular decomposition. Then the Cartan subalgebra  $H = \langle h_1, h_2 \rangle$  (where  $h_1 = E_{11} - E_{22}$ ,  $h_2 = E_{22} + E_{33}$ , and the  $E_{ij}$ 's are the  $2 \times 2$  matrix units). We use  $\epsilon_1, \epsilon_2$  and  $\delta_1$  to express the roots of  $G$  and choose  $\alpha = \epsilon_1 - \epsilon_2, \beta = \epsilon_2 - \delta_1$  as a simple root system. The positive even and odd roots are  $R_0^+ = \{\alpha\}, R_1^+ = \{\beta, \alpha + \beta\}$  respectively. Let  $\rho = -\beta$ . For  $\lambda \in H^*$ , let  $a = \lambda(h_1), b = \lambda(h_2)$ , and write  $\lambda = (a, b)$ . Note that

$$(3.1) \quad \alpha = (2, -1), \beta = (-1, 0), \rho = (1, 0).$$

By [5],  $\lambda \in H^*$  is typical if and only if  $(\lambda + \rho)(h_2) \neq 0$  and  $(\lambda + \rho)(h_1 + h_2) \neq 0$ , i.e, if and only if  $b \neq 0$  and  $a + b + 1 \neq 0$ . Let  $K(\lambda)$  be the Kac  $G$ -module with highest weight  $\lambda$  and let  $L(\lambda)$  be the simple  $G$ -module with highest weight  $\lambda$ . We call  $\lambda \in H^*$  dominant if  $L(\lambda)$  is finite dimensional and integral if  $\lambda = (a, b)$  with  $a, b \in \mathbb{Z}$ . If a dominant integral weight  $\lambda = (a, b)$  is atypical, then it is singly atypical with atypical root  $\beta$  or  $\alpha + \beta$ , and  $chK(\lambda) = chL(\lambda) + chL(\lambda - \beta)$  or  $chL(\lambda) + chL(\lambda - \alpha - \beta)$ . If  $v$  is a highest weight vector of  $K(\lambda)$ , then  $E_{32}v$  or  $(aE_{31} + E_{32}E_{21})v$  gives the other primitive vectors in  $K(\lambda)$ .

Let  $Q = \{m_1\alpha + m_2\beta : m_1, m_2 \in \mathbb{Z}\}$  and let  $Q^+ = \{m_1\alpha + m_2\beta : m_1, m_2 \in \mathbb{Z}_+\}$ . By identifying the elements of  $Q$  as pairs of integers as above, we identify  $Q$  as a subset of  $\mathbb{Z} \times \mathbb{Z}$ , and denote the elements of  $Q$  by  $(a, b)$  as before.

By a weight of  $U$ , we mean an element  $\omega = (\omega_1, \omega_2) \in \mathbb{C}(q)^{\times 2}$ . If  $\omega'$  is another weight, we write  $\omega' \leq \omega$  if  $\omega'_1^{-1}\omega_1 = q^a$  and  $\omega'_2^{-1}\omega_2 = q^b$  for some  $(a, b) \in Q^+$ . This induces a partial ordering on  $\mathbb{C}(q)^{\times 2}$  if  $q$  is not a root of 1. If  $V$  is a  $U$ -module, then its weight spaces are just the nonzero  $\mathbb{C}(q)$ -linear subspaces of the form  $V_\omega = \{v \in V : t_i v = \omega_i v, i = 1, 2\}$ . The nonzero elements in  $V_\omega$  are called weight vectors. A  $U$ -module is said to be a weight module if it is a direct sum of its weight subspaces. Note that these definitions also work for any  $\varepsilon \in \mathbb{C}^*$ .

A weight vector  $v$  is called maximal if  $U^+ \cdot v = 0$ . A highest weight  $U$ -module  $V$  is a  $U$ -module which contains a maximal vector  $v$  such that  $V = U \cdot v$ . For a weight  $\omega$ , one can define the Verma module  $V(\omega)$  by letting  $V(\omega) = U/J(\omega)$ , where  $J(\omega)$  is the left ideal of  $U$  generated by  $e_i$  and  $t_i - \omega_i, i = 1, 2$ . The module  $V(\omega)$  has a unique simple quotient  $L(\omega)$  and every simple highest weight  $U$ -module is isomorphic to some  $L(\omega)$ . One can also define the Kac module  $K(\omega)$  by first taking the simple highest weight  $U_0$ -module  $L_0(\omega)$  and extending it to a  $P$ -module by letting  $e_2$  act trivially on it, then setting  $K(\omega) = U \otimes_P L_0(\omega)$ . The unique simple quotient of  $K(\omega)$  is isomorphic to  $L(\omega)$ . For the roots of 1 case, one can also define the induced module similarly; we will provide more detail later. We first consider when the module  $L(\omega)$  is finite dimensional for the generic case.

Suppose that  $L(\omega)$  is finite dimensional (the discussion also works for  $U_\varepsilon$  if  $\varepsilon$  is not a root of 1). Then for a highest weight vector  $v$ ,  $U_0 \cdot v$  must be a finite dimensional  $U_0$ -module. So by the representation theory of the quantized enveloping algebras of Lie algebras,  $U_0 \cdot v \cong L_0(\omega)$ . Hence by the results of [8], if  $\omega = (\omega_1, \omega_2)$ , then  $\omega_1 = \pm q^a$  for some  $a \in \mathbb{Z}_+$ . On the other hand, if  $\omega_1 = \pm q^a$  for some  $a \in \mathbb{Z}_+$ , then  $L_0(\omega)$  is finite dimensional, and by the PBW theorem,  $K(\omega)$  is finite dimensional, and so is  $L(\omega)$ . Thus we have the following:

**Proposition 3.1.** *For  $\omega = (\omega_1, \omega_2)$ , the  $U$ -module  $L(\omega)$  is finite dimensional if and only if  $\omega_1 = \pm q^a$  for some  $a \in \mathbb{Z}_+$ .*

For  $0 \neq s \in \mathbb{C}(q)$ ,  $k \in \mathbb{Z}$ , let

$$[s; k] = (sq^k - s^{-1}q^{-k})/(q - q^{-1}).$$

For  $n \in \mathbb{N}$ , let

$$[n]! = [n][n-1] \cdots [1].$$

For a weight  $\omega = (\varepsilon q^a, \omega_2)$ ,  $\varepsilon = \pm 1$ ,  $a \in \mathbb{Z}_+$ , by the representation theory of  $U_0$ , the  $U_0$ -module  $L_0(\omega)$  is of dimension  $a+1$ , and if  $v_0$  is a highest weight vector,  $v_k = [k]!^{-1} f_1^k v_0$  ( $0 \leq k \leq a$ ) form a basis of  $L_0(\omega)$  such that (cf. (3.1))

$$(3.2) \quad t_1 v_k = \varepsilon q^{a-2k} v_k, \quad t_2 v_k = \omega_2 q^k v_k,$$

$$e_1 v_k = [\varepsilon q^a; 1-k] v_{k-1}, \quad f_1 v_k = [k+1] v_{k+1}.$$

Thus, upon writing the element  $u \otimes v$  as  $uv$  in  $K(\omega)$ , we see that  $K(\omega)$  has a basis  $\{v_k, f_2 v_k, f_3 v_k, f_3 f_2 v_k\}_{0 \leq k \leq a}$  such that

$$(3.3) \quad t_1 f_2 v_k = \varepsilon q^{a-2k+1} f_2 v_k, \quad t_1 f_3 v_k = \varepsilon q^{a-2k-1} f_3 v_k,$$

$$t_1 f_3 f_2 v_k = \varepsilon q^{a-2k} f_3 f_2 v_k;$$

$$t_2 f_2 v_k = \omega_2 q^k f_2 v_k,$$

$$t_2 f_3 v_k = \omega_2 q^{k+1} f_3 v_k, \quad t_2 f_3 f_2 v_k = \omega_2 q^{k+1} f_3 f_2 v_k.$$

Now let  $\ell > 2$  be an integer; let  $d = \ell$  if  $\ell$  is odd and  $d = \ell/2$  if  $\ell$  is even. Let  $\varepsilon$  be a primitive  $\ell$ 'th root of 1, and let  $U_\varepsilon, U_{\varepsilon,0}$  and so on, be the algebras defined as in section 2. Then by the representation theory of the quantized enveloping algebras of Lie algebras at the roots of 1, any finite dimensional simple weight module  $L_0$  for  $U_{\varepsilon,0}$  has dimension  $\leq d$ . If the dimension is  $< d$ , then the structure of  $L_0$  is similar to those  $L_0(\omega)$  for a generic  $q$  with a basis as described by (3.2) (where  $q$  is replaced by  $\varepsilon$ ). The  $d$ -dimensional ones can be described as follows: Every simple  $d$ -dimensional  $U_{\varepsilon,0}$ -module has a basis  $v_0, v_1, \dots, v_{d-1}$  such that for some  $\lambda_1, \lambda_2 \in \mathbb{C}^\times$ ,  $a, b \in \mathbb{C}$ ,

$$(3.4) \quad t_1 v_j = \lambda_1 \varepsilon^{-2j} v_j, \quad t_2 v_j = \lambda_2 \varepsilon^j v_j,$$

$$f_1 v_j = v_{j+1}, \quad j < d-1, \quad f_1 v_{d-1} = b v_0,$$

$$e_1 v_j = \left( \frac{(\lambda_1 \varepsilon^{1-j} - \lambda_1^{-1} \varepsilon^{j-1})(\varepsilon^j - \varepsilon^{-j})}{(\varepsilon - \varepsilon^{-1})^2} + ab \right) v_{j-1}, \quad j > 0, \quad e_1 v_0 = a v_{d-1}.$$

We denote these simple  $U_{\varepsilon,0}$ -modules by  $L_0(\lambda; a, b)$ , where  $\lambda = (\lambda_1, \lambda_2)$ ; we denote the induced module

$$U_{\varepsilon} \otimes_{P_{\varepsilon,0}} L_0(\lambda; a, b)$$

by  $K(\lambda; a, b)$ , where the action of  $P_{\varepsilon,0}$  on  $L_0(\lambda; a, b)$  is defined as usual. We also let

$$K(\lambda) = U_{\varepsilon} \otimes_{P_{\varepsilon,0}} L_0(\lambda)$$

for those  $L_0(\lambda)$  such that  $\dim L_0(\lambda) < d$ .

The module  $K(\lambda; a, b)$  (resp.  $K(\lambda)$ ) has a unique maximal submodule  $K'$  such that  $K(\lambda; a, b)/K'$  (resp.  $K(\lambda)/K'$ ) is simple. We let the simple quotient of  $K(\lambda; a, b)$  be  $L(\lambda; a, b)$  (resp.  $L(\lambda)$ ).

To simplify our discussion, we let  $\mathcal{U}$  be the algebra  $U$  or  $U_{\varepsilon}$ , and let the corresponding subalgebras be  $\mathcal{U}^+, \mathcal{U}^-, \mathcal{U}_0, \mathcal{P}$  and so on. We have the following lemma.

**Lemma 3.2.** *Let  $L$  be a finite dimensional simple weight  $\mathcal{U}$ -module. Then there exists a simple  $\mathcal{U}_0$ -submodule  $L_0 \subset L$  such that  $e_2 L_0 = 0$  and  $e_3 L_0 = 0$ . Therefore  $L$  is a quotient of the induced module  $\mathcal{U} \otimes_{\mathcal{P}} L_0$ .*

*Proof.* We may assume that  $L$  is not a quotient module of an induced module induced from a 0-dimensional  $\mathcal{U}_0$ -module, otherwise there is nothing to prove. We define subspaces  $L_i$  ( $i = 1, 2, 3$ ) of  $L$  by letting

$$L_1 = e_2 L, \quad L_2 = e_3 L, \quad L_3 = e_3 e_2 L,$$

and discuss the cases.

- i)  $L_3 \neq 0$ . We note that  $L_3$  is a  $\mathcal{U}_0$ -submodule of  $L$ . Since by lemma 2.1 and Lemma 2.2, we have

$$e_1 e_3 e_2 = e_3 e_2 e_1, \quad f_1 e_3 e_2 = e_3 e_2 f_1.$$

Now if  $L_0 \neq 0$  is a simple  $\mathcal{U}_0$ -submodule in  $L_3$ , then  $e_2 L_0 = 0, e_3 L_0 = 0$ . Moreover, since  $L$  is simple,  $L$  must be a quotient of the module induced by this  $L_0$ .

- ii)  $L_3 = 0, L_1 \neq 0$ . In this case  $S = L_1 + L_2$  is a  $\mathcal{U}_0$ -submodule in  $L$  (use Lemma 2.1 and Lemma 2.2 again), and  $e_2 S = 0, e_3 S = 0$ . Thus we can take  $L_0$  to be a nontrivial simple  $\mathcal{U}_0$ -submodule in  $S$ .
- iii)  $L_1 = 0$ . In this case, we can just take  $L_0$  to be a nontrivial simple  $\mathcal{U}_0$ -submodule in  $L$ .

The proof of the lemma is now complete. □

By Lemma 3.2, to study the finite dimensional simple weight  $\mathcal{U}$ -modules, one only needs to study the induced module  $\mathcal{U} \otimes_{\mathcal{P}} L_0$ , where  $L_0$  is a simple weight  $\mathcal{U}_0$ -module. The following proposition provides some sufficient conditions for the induced modules  $\mathcal{U} \otimes_{\mathcal{P}} L_0$  to be simple.

**Proposition 3.3.** *Let  $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}(q)^{\times 2}$  (or  $\mathbb{C}^{\times 2}$ ). Let  $K$  be the induced  $\mathcal{U}$ -module  $K(\lambda)$  (or  $K(\lambda; a, b)$  for some  $a, b \in \mathbb{C}$ ).*

- i) *If  $K$  is induced from a highest weight  $\mathcal{U}_0$ -module or  $ab = 0$ , then  $K$  is simple if  $\lambda_2 \neq \pm 1$  and  $\lambda_1 \lambda_2 \neq \pm q^{-1}$  (or  $\neq \pm \varepsilon^{-1}$ ).*
- ii) *The modules  $K(\lambda; a, b)$  with  $ab \neq 0$  are simple if*

$$[\lambda_2; 0]_{\varepsilon} [\lambda_1 \lambda_2; 1]_{\varepsilon} - ab \neq 0.$$

*Proof.* To avoid repetition, we assume that  $q$  can take the value  $\varepsilon$  in the proof.

Suppose that  $K$  is induced by the simple  $\mathcal{U}_0$ -module  $L_0$ . Let  $v_0 \in L_0$  be defined as before (see (3.2) or (3.4)). Then any nontrivial submodule of  $K$  contains the vector  $f_3 f_2 v_0$ . Hence if  $e_2 e_3 f_3 f_2 v_0 \neq 0$ , then  $K$  is simple. Since  $e_2 L_0 = 0$  and  $e_3 L_0 = 0$ , by Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} e_2 e_3 f_3 f_2 v_0 &= e_2(-f_3 e_3 + t_2 H_1 + t_1^{-1} H_2) f_2 v_0 \\ &= -e_2 f_3 e_3 f_2 v_0 + e_2 t_2 H_1 f_2 v_0 + e_2 t_1^{-1} H_2 f_2 v_0 \\ &= -e_2 f_3 (q t_2 e_1 - f_2 e_3) v_0 + [\lambda_1; 1] \lambda_2 e_2 f_2 v_0 + [\lambda_2; 0] \lambda_1^{-1} q^{-1} e_2 f_2 v_0 \\ &= -\lambda_2 (t_2^{-1} f_1 - f_3 e_2) e_1 v_0 + [\lambda_2; 0] ([\lambda_1; 1] \lambda_2 + [\lambda_2; 0] \lambda_1^{-1} q^{-1}) v_0 \\ &= -f_1 e_1 v_0 + [\lambda_2; 0] ([\lambda_1; 1] \lambda_2 + [\lambda_2; 0] \lambda_1^{-1} q^{-1}) v_0 \\ &= -f_1 e_1 v_0 + [\lambda_2; 0] [\lambda_1 \lambda_2; 1] v_0. \end{aligned}$$

If  $L_0$  is a highest weight  $\mathcal{U}_0$ -module, then

$$e_2 e_3 f_3 f_2 v_0 = [\lambda_2; 0] [\lambda_1 \lambda_2; 1] v_0.$$

So  $e_2 e_3 f_3 f_2 v_0 = 0$  if and only if

$$[\lambda_2; 0] = 0 \quad \text{or} \quad [\lambda_1 \lambda_2; 1] = 0,$$

that is,  $\lambda_2 = \pm 1$  or  $\lambda_1 \lambda_2 = \pm q^{-1}$ . If  $L_0 = L_0(\lambda; a, b)$ , then

$$e_2 e_3 f_3 f_2 v_0 = ([\lambda_2; 0] [\lambda_1 \lambda_2; 1] - ab) v_0.$$

Thus the proposition follows as desired.  $\square$

We will discuss the structures of the induced  $\mathcal{U}$ -modules  $K$  and their simple quotients for the generic case and the roots of 1 case separately in the next two sections.

#### 4. STRUCTURES OF FINITE DIMENSIONAL $K(\lambda)$ AND $L(\lambda)$ :

##### THE GENERIC CASE

Let  $q$  be an indeterminate or a nonzero element in  $\mathbb{C}^\times$  which is not a root of 1. According to Proposition 3.1 and Lemma 3.2, any finite dimensional simple weight  $U$ -module is a quotient of some  $K(\lambda)$  with  $\lambda = (\varepsilon q^a, \lambda_2)$ , where  $\varepsilon = \pm 1$  and  $a \in \mathbb{Z}_+$ . The following theorem provides a detailed description for the structures of the finite dimensional  $U$ -modules  $K(\lambda)$ , from which the structures of the finite dimensional  $U$ -modules  $L(\lambda)$  are also clear.

**Theorem 4.1.** *Let  $\lambda = (\varepsilon q^a, \lambda_2)$  with  $a \in \mathbb{Z}_+$ ,  $\lambda_2 \in \mathbb{C}(q)^\times$ . Then the  $U$ -module  $K(\lambda)$  is simple if and only if  $\lambda_2 \neq \pm 1$  and  $\lambda_2 \neq \pm q^{-a-1}$ . If  $\lambda_2 = \pm 1$ , the unique maximal submodule of  $K(\lambda)$  is isomorphic to  $L(\omega)$  with  $\omega = (\varepsilon q^{a+1}, \pm 1)$ . If  $\lambda_2 = \pm q^{-a-1}$ , then the unique maximal submodule of  $K(\lambda)$  is isomorphic to  $L(\omega)$  with:*

- 1)  $\omega = (\varepsilon q^{a-1}, \pm q^{-a})$  if  $a > 0$ ; or
- 2)  $\omega = (\varepsilon, \pm q^{-1})$  if  $a = 0$ .

*Proof.* By Proposition 3.2, we only need to consider the cases  $\lambda_2 = \pm 1$  or  $\pm q^{-a-1}$ .

If  $\lambda_2 = \pm 1$ , then since  $e_1 f_2 v_0 = 0$  and  $e_2 f_2 v_0 = [\lambda_2; 0] v_0 = 0$ ,  $f_2 v_0$  is a maximal vector in  $K(\lambda)$ . Furthermore, the submodule  $(f_2 v_0)$  generated by  $f_2 v_0$  is isomorphic to  $L(\omega)$  with  $\omega = (\varepsilon q^{a+1}, \pm 1)$  and  $L/(f_2 v_0)$  is simple. Hence the statement for this case follows.

If  $\lambda_2 = \pm q^{-a-1}$ , we let  $v = q^a f_2 v_1 + [a] f_3 v_0$  if  $a > 0$ , and let  $v = f_3 f_2 v_0$  if  $a = 0$ . Then one can verify that  $v$  is a maximal vector in  $K(\lambda)$ , the submodule  $(v)$  generated by  $v$  is isomorphic to  $L(\omega)$  with  $\omega = (\epsilon q^{a-1}, \pm q^{-a})$  or  $(-1, \pm q^{-1})$  respectively, and  $L/(v)$  is simple. Therefore the theorem follows as desired.  $\square$

From Theorem 4.1, we see that every simple finite dimensional highest weight module for  $U(sl(2, 1))$  admits a  $q$ -deformation in exactly the same sense as in the Lie algebra case: if  $a \in \mathbb{Z}_+, b \in \mathbb{Z}$ , then the  $U$ -module  $L((q^a, q^b))$  is the type 1  $q$ -deformation of the  $U(sl(2, 1))$ -module  $L((a, b))$ .

5. STRUCTURES OF FINITE DIMENSIONAL SIMPLE WEIGHT MODULES:  
THE ROOTS OF 1 CASE

Let  $\epsilon$  be a primitive  $\ell$ 'th root of 1 with  $\ell > 2$ . Define  $d, U_\epsilon, U_{\epsilon,0}$  and  $P_{\epsilon,0}$  as in section 3. Recall that the finite dimensional simple weight  $U_{\epsilon,0}$ -modules have dimensions  $\leq d$ , with the  $d$ -dimensional ones described by (3.4) and the  $r$ -dimensional ones ( $r < d$ ) given as follows: each one of them has a basis  $v_0, \dots, v_{r-1}$  such that (we omit the subscript  $\epsilon$  for  $[m]_\epsilon$ )

$$(5.1) \quad \begin{aligned} t_1 v_j &= \epsilon \epsilon^{r-2j} v_j, & t_2 v_j &= \lambda_2 \epsilon^j v_j, \\ e_1 v_j &= \epsilon[r-j+1] v_{j-1}, & f_1 v_j &= [j+1] v_{j+1}. \end{aligned}$$

By Proposition 3.2, to study the structures of the induced modules  $K(\lambda)$  or  $K(\lambda; a, b)$ , we only need to consider the following cases:

- 1)  $\lambda_2 = \pm 1$  or  $\lambda_1 \lambda_2 = \pm \epsilon^{-1}$ , for  $K(\lambda)$  or  $K(\lambda; a, b)$  with  $ab = 0$ ;
- 2)  $[\lambda_2; 0][\lambda_1 \lambda_2; 1] = ab$ , for  $K(\lambda; a, b)$  with  $ab \neq 0$ .

Let  $L_0$  be a finite dimensional simple weight  $U_{\epsilon,0}$ -module, let  $\dim L_0 = r$ , and let  $K = U_\epsilon \otimes_{P_{\epsilon,0}} L_0$ . For  $r < d, L_0 = L_0(\lambda)$  with  $\lambda = (\epsilon \epsilon^r, \lambda_2) \in \mathbb{C}^{\times 2}$ , the structure of  $K$  is similar to the generic case, and the proof is similar to the proof of Theorem 4.1. We omit the proof here.

If  $r = d$ , then  $L_0 = L_0(\lambda; a, b)$ . Let the notation be as in (3.4). Since  $L_0$  is simple, without loss of generality, we may assume that  $e_1 v_j \neq 0$  for  $0 < j \leq d-1$ . We now discuss different cases.

- (1)  $ab \neq 0$ . We only need to consider the case when

$$[\lambda_2; 0][\lambda_1 \lambda_2; 1] = ab.$$

Let  $M \neq 0$  be an arbitrary submodule of  $K$ ; then  $f_3 f_2 v_0 \in M$ . Note that we always have

$$(5.2) \quad \begin{aligned} e_2 f_3 f_2 v_0 &= (t_2^{-1} f_1 - f_3 e_2) f_2 v_0 \\ &= t_2^{-1} f_1 f_2 v_0 - f_3 H_2 v_0 \\ &= t_2^{-1} (\epsilon^{-1} f_2 f_1 - f_3) v_0 - [\lambda_2; 0] f_3 v_0 \\ &= \lambda_2^{-1} \epsilon^2 f_2 v_1 - (\lambda_2^{-1} \epsilon^{-1} + [\lambda_2; 0]) f_3 v_0 \\ &= \lambda_2^{-1} \epsilon^{-2} f_2 v_1 - \epsilon^{-1} [\lambda_2; 1] f_3 v_0, \end{aligned}$$

and

$$(5.3) \quad \begin{aligned} e_1 e_2 f_3 f_2 v_0 &= (\lambda_2^{-1} \epsilon^{-2} ([\lambda_1; 0] + ab) + \epsilon^{-1} [\lambda_2; 1] \lambda_1) f_2 v_0 - \epsilon^{-1} [\lambda_2; 1] a v_{d-1} \\ &= (\epsilon^{-1} [\lambda_1 \lambda_2; 1] + \lambda_2^{-1} \epsilon^{-2} ab) f_2 v_0 - \epsilon^{-1} [\lambda_2; 1] a v_{d-1}. \end{aligned}$$

If  $[\lambda_2; 1] \neq 0$ , i.e.  $\lambda_2 \neq \pm\varepsilon^{-1}$ , then by (5.3) we have

$$f_2e_1e_2f_3f_2v_0 = -\varepsilon^{-1}[\lambda_2; 1]af_2v_{d-1} \in M.$$

Therefore, since  $e_1v_j \neq 0$ ,  $e_1f_2 = f_2e_1$ ,  $f_2v_j \in M$  ( $0 \leq j \leq d-1$ ), and hence  $e_2f_2v_j = [\lambda_2; j]v_j \in M$ . Let  $0 \leq j \leq d-1$  be such that  $[\lambda_2; j] \neq 0$ . Then  $v_j \in M$ , and thus  $M = K$ ; i.e.  $K$  is simple.

If  $[\lambda_2; 1] = 0$ , then (5.2) implies that  $f_2v_1 \in M$ . Note that  $[\lambda_2; 1] = 0$  together with  $[\lambda_2; 0][\lambda_1\lambda_2; 1] = ab$  implies  $[\lambda_1; 0] + ab = 0$ . So the vector  $f_2v_1$  is a maximal vector in  $M$ , and in this case, the submodule  $(f_2v_1)$  of  $K$  generated by  $f_2v_1$  is the maximal submodule of  $K$ . By Lemma 2.3 and the following formula,

$$e_1f_1^{(k)} = [t_1; k-1]f_1^{(k-1)} + f_1^{(k)}e_1, \quad \text{where} \quad f_1^{(k)} = [k]!^{-1}f_1^k,$$

we see that  $K/(f_2v_1)$  has dimension  $2d$  with basis  $\{\bar{v}_k, f_3\bar{v}_k\}_{0 \leq k \leq d-1}$ .

(2)  $ab = 0$ . We separate this case into two subcases.

- (i)  $\lambda_2 = \pm 1$ . If  $a = 0$ , then  $f_2v_0$  is a maximal vector, and  $L(\lambda; a, b) = K/(f_2v_0)$  is of dimension  $2d$  with basis  $\{\bar{v}_k, f_3\bar{v}_k\}_{0 \leq k \leq d-1}$ . If  $a \neq 0$ , then by (5.3)

$$e_1e_2f_3f_2v_0 = \varepsilon^{-1}[\lambda_1\lambda_2; 1]f_2v_0.$$

If in addition  $[\lambda_1\lambda_2; 1] \neq 0$ , then  $f_2v_0 \in M$ . By applying  $e_1$  successively, we conclude that  $f_2v_j \in M$  ( $0 \leq j \leq d-1$ ) and hence  $M = K$ . That is,  $K$  is simple. If on the other hand  $[\lambda_1\lambda_2; 1] = 0$ , i.e.  $\lambda_1 = \pm\varepsilon^{-1}$ , then  $e_2f_3f_2v_0$  is maximal and  $L(\lambda; a, b) = K/(e_2f_3f_2v_0)$  is of dimension  $2d$ .

- (ii)  $\lambda_1\lambda_2 = \pm\varepsilon^{-1}$ ,  $\lambda_2 \neq \pm 1$ . In this case, using (5.3), we see that if  $a \neq 0$  and  $[\lambda_2; 1] \neq 0$ , then  $K$  is simple; and if at least one of  $a$  and  $[\lambda_2; 1]$  is 0, then  $e_2f_3f_2v_0$  is maximal and  $L(\lambda; a, b) = K/(e_2f_3f_2v_0)$  is of dimension  $2d$ .

To summarize, we have

**Theorem 5.1.** *Let the notation be as above.*

- i) *The structures of the modules  $K(\lambda)$  and  $L(\lambda)$  (where  $\lambda = (\varepsilon\varepsilon^r, \lambda_2)$  with  $r \in \mathbb{Z}_+$ ,  $\lambda_2 \in \mathbb{C}(q)$ ) are similar to the structures of the corresponding modules in the generic case. More precisely, the structures of these modules are described by Theorem 4.1.*
- ii) *The structures of the modules  $K(\lambda; a, b)$  and  $L(\lambda; a, b)$  can be described as follows. We have  $K(\lambda; a, b) = L(\lambda; a, b)$  except the following cases:*
  - (1)  $ab \neq 0$ ,  $[\lambda_2; 0][\lambda_1\lambda_2; 1] = ab$ ,  $\lambda_2 = \pm\varepsilon^{-1}$ ; then  $L(\lambda; a, b) = K(\lambda; a, b)/(f_2v_1)$ .
  - (2)  $a = 0$ ,  $\lambda_2 = \pm 1$ ; then  $L(\lambda; a, b) = K(\lambda; a, b)/(f_2v_0)$ .
  - (3)  $a = 0$ ,  $\lambda_2 \neq \pm 1$ ,  $\lambda_1\lambda_2 = \pm\varepsilon^{-1}$ ; then  $L(\lambda; a, b) = K(\lambda; a, b)/(e_2f_3f_2v_0)$ .
  - (4)  $b = 0$ ,  $a \neq 0$ ,  $\lambda_2 = \pm 1$ ,  $\lambda_1 = \pm\varepsilon^{-1}$ ; or  $b = 0$ ,  $a \neq 0$ ,  $\lambda_2 = \pm\varepsilon^{-1}$ ,  $\lambda_1 = \pm 1$ ; then  $L(\lambda; a, b) = K(\lambda; a, b)/(e_2f_3f_2v_0)$ .

Furthermore, for all four cases listed above,  $\dim L(\lambda; a, b) = 2d$ .

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