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WEIGHTED HARDY-LITTLEWOOD INEQUALITY FOR A-HARMONIC TENSORS

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Abstract. In this paper we prove a local weighted integral inequality for conjugate A-harmonic tensors similar to the Hardy and Littlewood integral inequality for conjugate harmonic functions. Then by using the local weighted integral inequality, we prove a global weighted integral inequality for conjugate A-harmonic tensors in John domains.

1. Introduction and notation

Conjugate harmonic functions have wide applications in many fields, such as potential theory, harmonic analysis and the theory of H^p-spaces. Conjugate A-harmonic tensors are interesting and important generalizations of conjugate harmonic functions and p-harmonic functions, $p > 1$. Many interesting results of conjugate A-harmonic tensors and their applications in fields such as quasiregular mappings and the theory of elasticity have been found recently, see [N3], [I], [IM], [S], [B] and [BM]. In this paper, we prove local weighted inequalities and global weighted integral inequalities for conjugate A-harmonic tensors in John domains.

Let $e_1, e_2, \cdots, e_n$ denote the standard unit basis of $\mathbf{R}^n$. For $l = 0, 1, \cdots, n$, the linear space of l-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \cdots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, is denoted by $\wedge^l = \wedge^l(\mathbf{R}^n)$. The Grassmann algebra $\wedge = \oplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by

$$\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$$

with summation over all l-tuples $I = (i_1, i_2, \cdots, i_l)$ and all integers $l = 0, 1, \cdots, n$. We define the Hodge star operator $\star$: $\wedge \to \wedge$ by the rule

$$\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n \quad \text{and} \quad \alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle (\star 1)$$

for all α, $\beta \in \wedge$.

Hence the norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \wedge^0 = \mathbf{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $\star : \wedge^l \to \wedge^{n-l}$ and $\star\star\,(-1)^{l(n-l)} : \wedge^l \to \wedge^l$.

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Throughout this paper, we always assume Ω is a connected open subset of $\mathbf{R}^n$. We write $\mathbf{R} = \mathbf{R}^1$. Cubes or balls are denoted by Q and σQ is the cube or ball with the same center as Q and with $diam(\sigma Q) = \sigma diam(Q)$. The n-dimensional Lebesgue measure of a set $E \subseteq \mathbf{R}^n$ is denoted by $|E|$. Suppose that $w \in L^1_{loc}(\mathbf{R}^n)$, $w > 0$ a.e., and $0 < p < \infty$. We denote the weighted L^p-norm of a measurable function f over E by

$$||f||_{p,E,w} = \left(\int_E |f(x)|^p w(x)dx\right)^{1/p} .$$

A differential l-form ω on Ω is a Schwartz distribution on Ω with values in $\wedge^l(\mathbf{R}^n)$. We denote the space of differential l-forms by $D'(\Omega, \wedge^l)$. We write $L^p(\Omega, \wedge^l)$ for the l-forms $\omega(x) = \sum_I \omega_I(x)dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ with $\omega_I \in L^p(\Omega, \mathbf{R})$ for all ordered l-tuples I. Thus $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$||\omega||_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx\right)^{1/p} = \left(\int_\Omega (\sum_I |\omega_I(x)|^2)^{p/2} dx\right)^{1/p} .$$

Similarly, $W^1_p(\Omega, \wedge^l)$ are those differential l-forms on Ω whose coefficients are in $W^1_p(\Omega, \mathbf{R})$. The notations $W^1_{p,loc}(\Omega, \mathbf{R})$ and $W^1_{p,loc}(\Omega, \wedge^l)$ are self-explanatory. We denote the exterior derivative by $d: D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \cdots, n$. Its formal adjoint operator $d^\star : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d\star$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \cdots, n$.

Definition 1.1. We call u a p-harmonic function if u satisfies the p-harmonic equation

$$\operatorname{div}(\nabla u|\nabla u|^{p-2}) = 0 \tag{1.2}$$

with $p > 1$. Its conjugate in the plane is a q-harmonic function v, $p^{-1} + q^{-1} = 1$, which satisfies

$$\nabla u|\nabla u|^{p-2} = (\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}).$$

Note that if $p = q = 2$, we get the usual conjugate harmonic funtions. If $\omega : \Omega \to \wedge^l$, then the value of $\omega(x)$ at the vectors $\xi_1, \cdots, \xi_l \in \mathbf{R}^n$ will be denoted by $\omega(x; \xi_1, \cdots, \xi_l)$. The following lemma appears in [IL].

Lemma 1.3. *Let $Q \subset \mathbf{R}^n$ be a cube. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty(Q, \wedge^l) \to C^\infty(Q, \wedge^{l-1})$ defined by*

$$(K_y\omega)(x; \xi_1, \cdots, \xi_l) = \int_0^1 t^{l-1}\omega(tx + y - ty; x - y, \xi_1, \cdots, \xi_{l-1})dt$$

and the decomposition

$$\omega = d(K_y\omega) + K_y(d\omega).$$

We define another linear operator $T_Q : C^\infty(Q, \wedge^l) \to C^\infty(Q, \wedge^{l-1})$ by averaging K_y over points y in Q

$$T_Q\omega = \int_Q \varphi(y)K_y\omega dy \ , \tag{1.4}$$

where $\varphi \in C_0^\infty(Q)$ is normalized by $\int_Q \varphi(y)dy = 1$. We define the l-form $\omega_Q \in D'(Q, \wedge^l)$ by

$$\omega_Q = |Q|^{-1} \int_Q \omega(y)dy, \quad l = 0, \text{ and } \omega_Q = d(T_Q\omega), \quad l = 1, 2, \cdots, n, \tag{1.5}$$

for all $\omega \in L^p(Q, \wedge^l)$, $1 \le p < \infty$.

In recent years there has been new interest developed in the study of the A-harmonic equation for differential forms, largely pertaining to applications in quasiconformal analysis and nonlinear elasticity, that is:

$$d^\star A(x, d\omega) = 0 \ , \tag{1.6}$$

where $A : \Omega \times \wedge^l(\mathbf{R}^n) \to \wedge^l(\mathbf{R}^n)$ satisfies the following conditions:

$$|A(x, \xi)| \le a|\xi|^{p-1} \text{ and } \langle A(x, \xi), \xi\rangle \ \ge \ |\xi|^p \tag{1.7}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbf{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.6). A solution to (1.6) is an element of the Sobolev space $W^1_{p,loc}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi\rangle = 0$$

for all $\varphi \in W^1_p(\Omega, \wedge^{l-1})$ with compact support.

In order to formulate the Hardy-Littlewood type estimate it is required first of all that the equation is written in the form of a first order differential system:

$$A(x, du) = d^\star v \ . \tag{1.8}$$

In this way we obtain a pair (u, v) of $(l-1)$-form u and $(l+1)$-form v, called the conjugate A-harmonic fields. Example: $du = d^*v$ is an analogue of a Cauchy-Riemann system in $\mathbf{R}^n$. Clearly, the A-harmonic equation is not affected by adding a closed form to u and coclosed form to v. Therefore, any type of estimates between u and v must be modulo such forms. Suppose that u is a solution to (1.6) in Ω. Then by Poincaré's lemma, at least locally in a ball B, there exists a form $v \in W^1_q(B, \wedge^{l+1})$, $\frac{1}{p} + \frac{1}{q} = 1$, such that (1.8) holds.

Definition 1.9. When u and v satisfy (1.8) in Ω, and A^{-1} exists in Ω, we call u and v conjugate A-harmonic tensors in Ω.

Hardy and Littlewood in [HL] proved the following result.

Theorem A. *For each $p > 0$, there is a constant C such that*

$$\int_D |u - u(0)|^p dxdy \le C \int_D |v - v(0)|^p dxdy$$

for all analytic functions $f = u + iv$ in the unit disk D.

Craig A. Nolder proves the similar results about K-quasiregular mappings in [N1] and [N2]. Recently, Craig A. Nolder generalized the above result and proved the following important Theorem B and Theorem C about conjugate A-harmonic tensors [N3].

Theorem B. *Let u and v be conjugate A-harmonic tensors in $\Omega \subset \mathbf{R}^n$, $\sigma > 1$, and $0 < s, t < \infty$. Then there exists a constant C, independent of u and v, such that*

$$||u - u_Q||_{s,Q} \leq C|Q|^{\beta}||v - c_1||_{t,\sigma Q}^{q/p} \tag{1.10}$$

and

$$||v - v_Q||_{t,Q} \leq C|Q|^{-\beta p/q}||u - c_2||_{s,\sigma Q}^{p/q}$$

for all cubes Q with $\sigma Q \subset \Omega$. Here c_1 is any form in $W^1_{p,loc}(\Omega, \Lambda)$ with $d^\star c_1 = 0$, c_2 is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $dc_2 = 0$ and $\beta = 1/s + 1/n - (1/t + 1/n)q/p$.

Theorem C. *Let $u \in D^{'}(\Omega, \Lambda^0)$ and $v \in D^{'}(\Omega, \Lambda^2)$ be conjugate A-harmonic tensors and $0 < s, t < \infty$. If Ω is a δ-John domain, $q \leq p$, $v - c \in L^t(\Omega, \Lambda^2)$ and*

$$s = \Phi(t) = \frac{npt}{nq + t(q-p)}\ , \qquad 0 < t < \infty, \tag{1.11}$$

then $u \in L^s(\Omega, \Lambda^0)$ and moreover, there exists a constant C, independent of u and v, such that

$$\| u - u_{Q_0} \|_{s,\Omega} \leq \ C \| v - c \|_{t,\Omega}^{q/p}\ . \tag{1.12}$$

Here c is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $d^*c = 0$.

Our main results Theorem 2.4 and Theorem 3.4 generalize (1.10) and (1.12), respectively.

2. The local weighted integral inequality

We will use the following generalized Hölder's inequality.

Lemma 2.1. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbf{R}^n$, then*

$$\| fg \|_{s,\Omega} \leq \| f \|_{\alpha,\Omega} \cdot \| g \|_{\beta,\Omega}$$

for any $\Omega \subset \mathbf{R}^n$.

The following definition appears in [G].

Definition 2.2. We say the weight $w(x) > 0$ satisfies the A_r condition, $r > 1$, write $w \in A_r$, if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w dx \right) \left(\frac{1}{|Q|} \int_Q \left(\frac{1}{w} \right)^{1/(r-1)} dx \right)^{r-1} < \infty$$

for any cube $Q \subset \mathbf{R}^n$.

We also need the following lemma [G].

Lemma 2.3. *If $w \in A_r$, $r > 1$, then there exist constants $\beta > 1$ and C, independent of w, such that*

$$\| w \|_{\beta,Q} \leq C|q|^{(1-\beta)/\beta} \| w \|_{1,Q}$$

for all cubes $Q \subset \mathbf{R}^n$.

Now, we can prove the following local weighted result.

Theorem 2.4. *Let u and v be conjugate A-harmonic tensors in a domain $\Omega \subset \mathbf{R}^n$ and $w \in A_r$. Let $s = \Phi(t)$ be defined by* (1.11). *Then there exists a constant C, independent of u and v, such that*

$$\left(\int_Q |u-u_Q|^s w dx\right)^{1/s} \le C\left(\int_{\sigma Q} |v-c|^t w^{pt/qs} dx\right)^{q/pt} \tag{2.5}$$

for all cubes Q with $\sigma Q \subset \Omega \subset \mathbf{R}^n$ and $\sigma > 1$. Here c is any form in $W^1_{q,loc}(\Omega,\Lambda)$ with $d^*c = 0$.

Proof. By Lemma 2.3, there exist constants $\alpha > 1$ and C_1, independent of w, such that

$$\| w \|_{\alpha,\sigma Q} \le C_1 |Q|^{(1-\alpha)/\alpha} \| w \|_{1,\sigma Q} . \tag{2.6}$$

Since $1/\alpha s + (\alpha-1)/\alpha s = 1/s$, then by Lemma 2.1, we have

$$\| u-u_Q \|_{s,Q,w} \le \| w \|^{1/s}_{\alpha,\ Q} \cdot \| u-u_Q \|_{\alpha s/(\alpha-1),\ Q} . \tag{2.7}$$

By Theorem B, there is a constant C_2, independent of u and v, such that for any $t' > 0$, we have

$$\| u-u_Q \|_{\alpha s/(\alpha-1),\ Q} \le C_2 |Q|^{\beta'} \cdot \| v-c \|^{q/p}_{t',\ \sigma Q} , \tag{2.8}$$

where $\beta' = (\alpha-1)/\alpha s + 1/n - (1/t' + 1/n)q/p$. Combining (2.7) and (2.8), we obtain

$$\| u-u_Q \|_{s,Q,w} \le C_2 |Q|^{\beta'} \cdot \| w \|^{1/s}_{\alpha,\ Q} \cdot \| v-c \|^{q/p}_{t',\ \sigma Q} . \tag{2.9}$$

Now, choose $t' = t/k$, where k is to be determined later, and note $|v-c| = w^{-p/qs}|v-c|w^{p/qs}$; by Lemma 2.1, we get

$$\| v-c \|_{t',\ \sigma Q} \le ||(1/w)^{pt/qs}||^{1/t}_{1/(k-1),\ \sigma Q} \cdot \left(\int_{\sigma Q} |v-c|^t w^{pt/qs} dx\right)^{1/t} . \tag{2.10}$$

From (2.6), (2.9) and (2.10) we have

$$\begin{aligned} \| u-u_Q \|_{s,Q,w} \le & C_3 |Q|^{\beta'+(1-\alpha)/\alpha s} \| w \|^{1/s}_{1,\sigma Q} \\ & \cdot ||(\frac{1}{w})^{pt/qs}||^{q/pt}_{1/(k-1),\ \sigma Q} \left(\int_{\sigma Q} |v-c|^t w^{pt/qs} dx\right)^{q/pt} . \end{aligned} \tag{2.11}$$

We choose $k = 1 + pt(r-1)/qs$. Then $(k-1)qs/pt = r-1$, and by $w \in A_r$, we know

$$\begin{aligned} & \| w \|^{1/s}_{1,\sigma Q} ||(\frac{1}{w})^{pt/qs}||^{q/pt}_{1/(k-1),\sigma Q} \\ & = |\sigma Q|^{1/s+(k-1)q/pt} \left(\frac{1}{|\sigma Q|}\int_{\sigma Q} w dx \left(\frac{1}{|\sigma Q|}\int_{\sigma Q} \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1}\right)^{1/s} \\ & \le C_4 \, |Q|^{1/s+(k-1)q/pt} . \end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we have

$$\| u-u_Q \|_{s,Q,w} \le C_5 \, |Q|^{\gamma} \left(\int_{\sigma Q} |v-c|^t w^{pt/qs} dx\right)^{q/pt} , \tag{2.13}$$

where $\gamma = \beta' + (1-\alpha)/\alpha s + 1/s + q(k-1)/pt = -(nq + t(q-p))/npt + 1/s = 0$ by (1.11). So (2.13) becomes

$$\| u - u_Q \|_{s,Q,w} \leq C \left(\int_{\sigma Q} |v-c|^t w^{pt/qs} dx \right)^{q/pt},$$

that is,

$$\left(\int_Q |u - u_Q|^s w dx \right)^{1/s} \leq C \left(\int_{\sigma Q} |v-c|^t w^{pt/qs} dx \right)^{q/pt}.$$

We have completed the proof of Theorem 2.4. □

3. The global weighted integral inequality

Definition 3.1. We call Ω, a proper subdomain of $\mathbf{R}^n$, a δ-John domain, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that

$$d(\xi, \partial\Omega) \geq \delta |x - \xi|$$

for each $\xi \in \gamma$. Here $d(\xi, \partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.

We know that a δ-John domain has the following properties [N3].

Lemma 3.2. *Let $\Omega \subset \mathbf{R}^n$ be a δ-John domain. Then there exists a covering $\mathcal{V}$ of Ω consisting of open cubes such that:*

i) $\sum_{Q \in \mathcal{V}} \chi_{\sigma Q}(x) \leq N \chi_\Omega(x), \quad x \in \mathbf{R}^n$.

ii) *There is a distinguished cube $Q_0 \in \mathcal{V}$ (called the central cube) which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \cdots, Q_k = Q$ from $\mathcal{V}$ such that for each $i = 0, 1, \cdots, k-1$,*

$$Q \subset N Q_i.$$

There is a cube $R_i \subset \mathbf{R}^n$ (this cube does not need be a member of $\mathcal{V}$) such that

$$R_i \subset Q_i \cap Q_{i+1}, \text{ and } Q_i \cup Q_{i+1} \subset N R_i \ .$$

The following lemma appears in [IN].

Lemma 3.3. *If $\mathcal{V}$ is a collection of cubes in $\mathbf{R}^n$ and C_Q are non-negative numbers associated with the cubes $Q \in \mathcal{V}$ and $w \in A_r$, $d\mu(x) = w(x)dx$, then for $1 \leq p < \infty$ and $N \geq 1$ we have*

$$\left(\int_{\mathbf{R}^n} (\sum_{Q \in \mathcal{V}} C_Q \cdot \chi_{NQ})^p d\mu(x) \right)^{1/p} \leq B_p \left(\int_{\mathbf{R}^n} (\sum_{Q \in \mathcal{V}} C_Q \cdot \chi_Q)^p d\mu(x) \right)^{1/p},$$

where B_p is independent of the collection $\mathcal{V}$ and the numbers C_Q.

Theorem 3.4. *Let $u \in D'(\Omega, \Lambda^0)$ and $v \in D'(\Omega, \Lambda^2)$ be conjugate A-harmonic tensors. Let $q \leq p$, $v - c \in L^t(\Omega, \Lambda^2)$, $w \in A_r$ and $s = \Phi(t)$ is defined in (1.11). Then there exists a constant C, independent of u and v, such that*

$$\left(\int_\Omega |u - u_{Q_0}|^s w dx \right)^{1/s} \leq C \left(\int_\Omega |v-c|^t w^{pt/qs} dx \right)^{q/pt}$$

for any δ-John domain $\Omega \subset \mathbf{R}^n$. Here c is any form in $W^1_{q,loc}(\Omega, \Lambda)$ with $d^*c = 0$ and $Q_0 \subset \Omega$ is the cube appearing in Lemma 3.2.

Proof. Since $w \in A_r$, we can write $d\mu(x) = w(x)dx$; then (2.5) can be written as

$$\int_Q |u - u_Q|^s d\mu(x) \le C \left(\int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{qs/pt}. \tag{3.5}$$

We use the notations and the covering $\mathcal{V}$ described in the above Lemma 3.2 and the properties of the measure $d\mu(x) = w(x)dx$: if $w \in A_r$, then

$$\mu(NQ) \le MN^{nr}\mu(Q) \tag{3.6}$$

for each cube Q with $NQ \subset \mathbf{R}^n$ (see [G]) and

$$\max(\mu(Q_i), \mu(Q_{i+1})) \le MN^{nr}\mu(Q_i \cap Q_{i+1}) \tag{3.7}$$

for the sequence of cubes $Q_i, Q_{i+1}, i = 0, 1, \cdots, k-1$ described in ii). We will use the elementary inequality $|a+b|^s \le 2^s(|a|^s + |b|^s)$ for all $s > 0$. In particular we have

(3.8)

$$\begin{aligned}\int_\Omega |u - u_{Q_0}|^s w dx &= \int_\Omega |u - u_{Q_0}|^s d\mu(x) \\ &\le 2^s \sum_{Q \in \mathcal{V}} \int_Q |u - u_Q|^s d\mu(x) + 2^s \sum_{Q \in \mathcal{V}} \int_Q |u_{Q_0} - u_Q|^s d\mu(x).\end{aligned}$$

The first sum can be estimated by (3.5) and the condition i):

$$\begin{aligned}\sum_{Q \in \mathcal{V}} \int_Q |u - u_Q|^s d\mu(x) &\le C_1 \sum_{Q \in \mathcal{V}} \left(\int_{\sigma Q} |v - c|^t w^{pt/qs} dx \right)^{qs/pt} \\ &\le C_1 N \left(\int_\Omega |v - c|^t w^{pt/qs} dx \right)^{qs/pt}.\end{aligned} \tag{3.9}$$

Now we estimate the second sum in (3.8). Fix a cube $Q \in \mathcal{V}$ and let $Q_0, Q_1, \cdots, Q_k = Q$ be the chain from ii). We have

$$|u_{Q_0} - u_Q| \le \sum_{i=0}^{k-1} |u_{Q_i} - u_{Q_{i+1}}|. \tag{3.10}$$

From (3.5) and (3.7) we have

$$\begin{aligned}|u_{Q_i} - u_{Q_{i+1}}|^s &= \frac{1}{\mu(Q_i \cap Q_{i+1})} \int_{Q_i \cap Q_{i+1}} |u_{Q_i} - u_{Q_{i+1}}|^s d\mu(x) \\ &\le \frac{MN^{nr}}{\max(\mu(Q_i), \mu(Q_{i+1}))} \int_{Q_i \cap Q_{i+1}} |u_{Q_i} - u_{Q_{i+1}}|^s d\mu(x) \\ &\le C_2 \sum_{j=i}^{i+1} \frac{1}{\mu(Q_j)} \int_{Q_j} |u - u_{Q_j}|^s d\mu(x) \\ &\le C_3 \sum_{j=i}^{i+1} \frac{1}{\mu(Q_j)} \left(\int_{\sigma Q_j} |v - c|^t w^{pt/qs} dx \right)^{qs/pt}\end{aligned}$$

Since $Q \subset NQ_j$ for $j = i, i+1, 0 \le i \le k-1$ (see ii)), we have

$$|u_{Q_i} - u_{Q_{i+1}}|^s \chi_Q(x) \le C_3 \sum_{j=i}^{i+1} \frac{\chi_{NQ_j}(x)}{\mu(Q_j)} \left(\int_{\sigma Q_j} |v - c|^t w^{pt/qs} dx \right)^{qs/pt}.$$

By (3.10) we have (note $|a+b|^{1/s} \le 2^{1/s}(|a|^{1/s}+|b|^{1/s})$)

$$|u_{Q_0}-u_Q|\chi_Q(x) \le C_4 \sum_{R\in\mathcal{V}} \left(\frac{1}{\mu(R)}\left(\int_{\sigma R}|v-c|^t w^{pt/qs}dx\right)^{qs/pt}\right)^{1/s}\cdot \chi_{NR}(x)$$

for every $x\in\mathbf{R}^n$. Hence

(3.11)

$$\sum_{Q\in\mathcal{V}}\int_Q |u_{Q_0}-u_Q|^s d\mu(x) \le C_5\int_{R^n}\left|\sum_{R\in\mathcal{V}}\left(\frac{1}{\mu(R)}\left(\int_{\sigma R}|v-c|^t w^{pt/qs}dx\right)^{qs/pt}\right)^{1/s}\chi_{NR}(x)\right|^s d\mu(x).$$

If $0\le s\le 1$, we use the inequality $|\sum t_\alpha|^s \le \sum |t_\alpha|^s$, (3.6) and the condition i) to get

$$\begin{aligned}\sum_{Q\in\mathcal{V}}\int_Q |u_{Q_0}-u_Q|^s d\mu(x) &\le C_6\sum_{R\in\mathcal{V}}\frac{\mu(NR)}{\mu(R)}\left(\int_{\sigma R}|v-c|^t w^{pt/qs}dx\right)^{qs/pt}\\ &\le C_7\sum_{R\in\mathcal{V}}\left(\int_{\sigma R}|v-c|^t w^{pt/qs}dx\right)^{qs/pt}.\end{aligned}$$

Note $qs/pt\ge 1$ and $\sum t_\alpha^p \le (\sum t_\alpha)^p$ for $p\ge 1$ and $t_\alpha>0$; then

$$\begin{aligned}\sum_{Q\in\mathcal{V}}\int_Q |u_{Q_0}-u_Q|^s d\mu(x) &\le C_7\sum_{R\in\mathcal{V}}\left(\int_\Omega |v-c|^t w^{pt/qs}\chi_{\sigma R}(x)dx\right)^{qs/pt}\\ &\le C_7\left(\int_\Omega |v-c|^t w^{pt/qs}\sum_{R\in\mathcal{V}}\chi_{\sigma R}(x)dx\right)^{qs/pt}\\ &\le C_7\left(\int_\Omega |v-c|^t w^{pt/qs}N\chi_\Omega(x)dx\right)^{qs/pt}\\ &\le C_8\left(\int_\Omega |v-c|^t w^{pt/qs}dx\right)^{qs/pt}.\end{aligned} \tag{3.12}$$

Combining (3.8), (3.9) and (3.12), we have proved our theorem for the case $0<s\le 1$.

For the case $1\le s<\infty$, by (3.11) and Lemma 3.3, we have

$$\sum_{Q\in\mathcal{V}}\int_Q |u_{Q_0}-u_Q|^s d\mu(x) \le C_9\int_{R^n}\left|\sum_{R\in\mathcal{V}}\left(\frac{1}{\mu(R)}\left(\int_{\sigma R}|v-c|^t w^{pt/qs}dx\right)^{qs/pt}\right)^{1/s}\chi_R(x)\right|^s d\mu(x).$$

Note

$$\sum_{R\in\mathcal{V}}\chi_R(x) \le \sum_{R\in\mathcal{V}}\chi_{\sigma R}(x) \le N\chi_\Omega(x)$$

with the elementary inequality $|\sum_{i=1}^N t_i|^s \leq N^{s-1}\sum_{i=1}^N |t_i|^s$; we obtain

$$\begin{aligned}
&\sum_{Q\in\mathcal{V}}\int_Q |u_{Q_0}-u_Q|^s d\mu(x)\\
&\quad\leq C_{10}\int_{R^n}\left(\sum_{R\in\mathcal{V}}\frac{1}{\mu(R)}\left(\int_{\sigma R}|v-c|^t w^{pt/qs}dx\right)^{qs/pt}\chi_R(x)\right)d\mu(x)\\
&\quad= C_{10}\sum_{R\in\mathcal{V}}\left(\int_{\sigma R}|v-c|^t w^{pt/qs}dx\right)^{qs/pt}\\
&\quad\leq C_{11}\left(\int_\Omega |v-c|^t w^{pt/qs}dx\right)^{qs/pt}
\end{aligned}\tag{3.13}$$

by the condition i). Combining (3.8), (3.9) and (3.13), we have proved the theorem for the case $1\leq s<\infty$; thus, we have completed the proof of Theorem 3.4. □

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