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A RELATION BETWEEN THE PLURICOMPLEX AND THE CLASSICAL GREEN FUNCTIONS IN THE UNIT BALL OF $\mathbf{C}^n$

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Abstract. We give a sharp upper bound for the quotient of the pluricomplex and the classical Green functions in the unit ball of $\mathbf{C}^n$.

1. Introduction

The classical Green function for the unit ball $B = B(0,1)$ in $\mathbf{C}^n$ or $\mathbf{R}^{2n}$ is

$$G_n(z,w) = \begin{cases} |w|^{2-2n} \left| z - w/|w|^2 \right|^{2-2n} - |z-w|^{2-2n} & \text{for } w \neq 0, \\ 1 - |z|^{2-2n} & \text{for } w = 0. \end{cases}$$

Here we defined it as a non-positive subharmonic function, whereas in most texts it is defined as the negative of our function (see for example [H], p. 77).

On the other hand, the pluricomplex Green function is $g_n(z,w) = \log |T_w(z)|$ where $T_w(z)$ is the Möbius transformation which maps w onto the origin. To be explicit,

$$T_w(z) = \frac{w - P_w(z) - \left(1 - |w|^2\right)^{1/2} Q_w(z)}{1 - \langle z, w \rangle}$$

where P_w is the orthogonal projection of $\mathbf{C}^n$ onto the subspace generated by w, and Q_w the projection onto the orthogonal complement of that subspace ([K], p. 148 and 224).

We define, for $n \geq 2$,

$$h_n(z,w) = \frac{g_n(z,w)}{G_n(z,w)}.$$

Note that h_n is a non-negative function, which can be extended to a continuous function on $B \times B$, since

$$\lim_{z,w \to \varsigma} h_n(z,w) = 0 \quad \text{for all } \varsigma \in B.$$

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2. The result

The following theorem was proved in [Ca].

Theorem 1. *For all* $z, w \in B(0,1) \subset \mathbf{C}^n$, $n \geq 2$, *the inequality*

$$h_n(z,w) < 2^{2n-3}/(n-1)$$

holds. Moreover, this estimate is the best possible.

Proof. In the case $w = 0$, h_n reduces to $\log|z|/(1-|z|^{2-2n})$. Let

$$f(x) = \frac{\log x}{1 - x^{2-2n}}$$

for all $x \in (0,1)$. Then differentiation gives

$$f'(x) = \frac{x^{2n-2} + (2-2n)\log x - 1}{x^{2n-1}(1-x^{2-2n})^2}.$$

By differentiating the numerator of $f'(x)$ we can easily see that it is decreasing and therefore greater than its value for $x = 1$, which is 0. Therefore $f(x)$ is increasing. Since $\lim_{x \to 1-} f(x) = 1/(2n-2) = 2^{-1}/(n-1) < 2^{2n-3}/(n-1)$, the theorem is proved in the case $w = 0$.

For the rest of the proof, assume $w \neq 0$. Let $\lambda = \langle z, w\rangle / |w|^2$. Then $P_w(z) = \lambda w$ and if we let $Q_w(z) = z^*$, then $z = \lambda w + z^*$. Hence

$$g_n(z,w) = \frac{1}{2}\log|T_w(z)|^2 =$$

$$= \frac{1}{2}\log\left|w - \lambda w - \left(1 - |w|^2\right)^{1/2} z^*\right|^2 - \frac{1}{2}\log|1 - \langle\lambda w + z^*, w\rangle|^2 =$$

$$= \frac{1}{2}\log\left(|1-\lambda|^2 |w|^2 + \left(1-|w|^2\right)|z^*|^2\right) - \frac{1}{2}\log\left|1 - \lambda|w|^2\right|^2$$

using the orthogonality between z^* and w. By a similar argument we have

$$G_n(z,w) = |w|^{2-2n}\left|\lambda w - w/|w|^2 + z^*\right|^{2-2n} - |(\lambda-1)w + z^*|^{2-2n} =$$

$$= |w|^{2-2n}\left(\left|\lambda w - w|w|^{-2}\right|^2 + |z^*|^2\right)^{1-n} - \left||\lambda-1|^2|w|^2 + |z^*|^2\right|^{1-n} =$$

$$= \left(\left|\lambda w - w|w|^{-2}\right|^2 |w|^2 + |z^*|^2|w|^2\right)^{1-n} - \left||\lambda-1|^2|w|^2 + |z^*|^2\right|^{1-n} =$$

$$= \left(\left|\lambda|w|^2 - 1\right|^2 + |z^*|^2|w|^2\right)^{1-n} - \left||\lambda-1|^2|w|^2 + |z^*|^2\right|^{1-n}.$$

To simplify the notation, let $x = |w|^2$, $u = |z^*|^2|w|^2$. After multiplying by $G_n < 0$ we can write the desired inequality in the following way

$$\frac{1}{2}\log\left(|1-\lambda|^2 x + (1-x)u/x\right) - \frac{1}{2}\log|1-\lambda x|^2 >$$

$$> \frac{2^{2n-3}/(n-1)}{\left(|1-\lambda x|^2 + u\right)^{n-1}} - \frac{2^{2n-3}/(n-1)}{\left(|1-\lambda|^2 x + u/x\right)^{n-1}}.$$

Since $1 > |z|^2 = |\lambda|^2 |w|^2 + |z^*|^2 = |\lambda|^2 x + u/x$, it suffices to show that this inequality holds for all $0 < x < 1$, $0 \le u < x$ and $|\lambda| < \sqrt{x-u}/x$. For each $u \in [0,1)$, define

$$f_{u,n}(t) = \log t + \frac{4^{n-1}/(n-1)}{(t+u)^{n-1}}.$$

Our inequality now takes the form

$$f_{u,n}\left(|1-\lambda|^2 x + u/x - u\right) > f_{u,n}\left(|1-\lambda x|^2\right).$$

Differentiation shows that $f_{u,n}(t)$ is decreasing when $(t+u)^n < 4^{n-1}t$. In the case $n = 2$, this occurs precisely when $t \in \left(2-u-2\sqrt{1-u}, 2-u+2\sqrt{1-u}\right)$. We claim that when t belongs to this interval, then $f_{u,n}(t)$ is decreasing for all n. Indeed, as $(t+u)^2 < 4t$ and $0 < t < 4$, we have $(t+u)^n = \left[(t+u)^2\right]^{n/2} < (4t)^{n/2} \le (4t)^{n/2}(4/t)^{n/2-1} = 4^{n-1}t$. Therefore it suffices to show that

$$2-u-2\sqrt{1-u} <_{(1)} |1-\lambda|^2 x + u/x - u <_{(2)}$$

$$<_{(2)} |1-\lambda x|^2 <_{(3)} 2-u+2\sqrt{1-u}$$

for all $0 < x < 1$, $0 \le u < x$ and $|\lambda| < \sqrt{x-u}/x$. We prove the three inequalities separately.

(1) $2-u-2\sqrt{1-u} < |1-\lambda|^2 x + u/x - u$

The proof is split into two cases.

Case 1. $\sqrt{x-u}/x \le 1$.

Since $|1-\lambda| \ge 1-|\lambda|$, it is enough to prove (1) when λ is real and non-negative. Moreover, $0 \le \lambda < \sqrt{x-u}/x$ for such λ, so it suffices to show (1) when $\lambda = \sqrt{x-u}/x$, i.e. to show that $\left(1-\sqrt{x-u}/x\right)^2 x + u/x > 2-2\sqrt{1-u}$. Expanding the square and simplifying, we see that this is equivalent to $2\sqrt{1-u} > 1+2\sqrt{x-u}-x$. Both sides of this inequality are non-negative, so it is enough to prove the squared inequality, namely (after simplifying) $x+3 > 4\sqrt{x-u}$. It is easy to check that this is true, even for $u = 0$.

Case 2. $\sqrt{x-u}/x > 1$, i.e. $u < x - x^2$.

In this case $|1-\lambda|^2 x$ can be zero, so we have to show the inequality without that term, i.e. we must prove that $u/x > 2-2\sqrt{1-u}$, or equivalently $u/\left(2-2\sqrt{1-u}\right) > x$. If we view the left side of this inequality as a function of u, we can easily see, by differentiation, that it is decreasing and therefore greater than its value in $u = x - x^2$ which is $\left(x-x^2\right)/\left(2-2\sqrt{1-x+x^2}\right)$. Elementary calculation shows that this expression is greater than or equal to x, and (1) is proved.

(2) $|1-\lambda|^2 x + u/x - u < |1-\lambda x|^2$

By using the bound $|\lambda| < \sqrt{x-u}/x$ we obtain

$$|1-\lambda|^2 x + \frac{u}{x} - u < (1+|\lambda|^2 - 2\mathrm{Re}(\lambda))x + \left(\frac{1}{x}-1\right)(x - |\lambda|^2 x^2) =$$

$$= 1 + |\lambda|^2 x^2 - 2\mathrm{Re}(\lambda)x = |1-\lambda x|^2.$$

(3) $|1-\lambda x|^2 < 2-u+2\sqrt{1-u}$

The same bound as used in the proof of (2) gives

$$|1-\lambda x|^2 \le (1+|\lambda x|)^2 \le \left(1+\sqrt{x-u}\right)^2 = 1+x-u+2\sqrt{x-u}.$$

Therefore

$$|1-\lambda x|^2 - \left(2-u+2\sqrt{1-u}\right) \leq$$

$$\leq 1+x-u+2\sqrt{x-u} - \left(2-u+2\sqrt{1-u}\right) =$$

$$= x-1+2\left(\sqrt{x-u}-\sqrt{1-u}\right) < 0$$

since $x<1$. This completes the proof of (3).

It remains to show that the estimate is sharp. Let $w=(t,0,...,0)$ and $z=(-t,0,...,0)$, where t is real and positive. Then $\lambda=-1$ and $z^*=0$. We substitute this in the expression for h_n. Then using Taylor's formula near $t=1$ and the identity $a^m-b^m=(a-b)\left(a^{m-1}+a^{m-2}b+...+b^{m-1}\right)$ we get

$$h_n(z,w) = \frac{\log(2t)-\log(1+t^2)}{(1+t^2)^{2-2n}-(2t)^{2-2n}} =$$

$$= -\frac{(2t)^{2n-2}(1+t^2)^{2n-2}}{\left((2t)^2\right)^{n-2}+\left((2t)^2\right)^{n-3}(1+t^2)^2+...+\left((1+t^2)^2\right)^{n-2}} \times$$

$$\times \frac{\log 2+(t-1)-(t-1)^2/2+O(t-1)^3-(\log 2+(t-1)+O(t-1)^3)}{(1-t)^2(1+t)^2} =$$

$$= -\frac{(2t)^{2n-2}(1+t^2)^{2n-2}}{\left((2t)^2\right)^{n-2}+\left((2t)^2\right)^{n-3}(1+t^2)^2+...+\left((1+t^2)^2\right)^{n-2}} \times$$

$$\times \left[-\frac{1}{2}+O(t-1)\right]\frac{1}{(1+t)^2} \to$$

$$\to -\frac{2^{2n-2}\cdot 2^{2n-2}}{2^2\left[\left(2^2\right)^{n-2}(n-1)\right]} \cdot \left(-\frac{1}{2}\right) = \frac{2^{2n-3}}{n-1}$$

when $t\to 1$. ☐

Remark. The example at the end of the proof shows that $h_n(z,w)$ tends to its supremum, when z and w approach opposite points at the boundary. Since also $h_n(z,w)$ has its least value 0 for $z=w$, we conjecture that

$$\sup_{|z|\leq|w|} h_n(z,w) = h_n(-w,w)$$

for all $w\in B$.

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