PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 6, June 1997, Pages 1835–1839
S 0002-9939(97)03943-9

ESTIMATING A SKEIN MODULE WITH $SL_2(\mathbb{C})$ CHARACTERS

DOUG BULLOCK

(Communicated by Ronald Stern)

ABSTRACT. We introduce a new technique for estimating the number of generators of the Kauffman bracket skein module of a three manifold; one which requires the construction of linear functionals on a simpler version of the module. Of particular interest is the use of representations of the fundamental group into $SL_2(\mathbb{C})$ to generate the functionals.

1. INTRODUCTION AND DEFINITIONS

The motivation behind this work is a pattern of behavior demonstrated by all the known examples [2, 3, 5, 6, 8] of Kauffman bracket skein modules of compact orientable 3-manifolds. Namely, the module is finitely generated if and only if the underlying manifold does not contain an incompressible surface. For most of these examples the module has a well understood presentation, but for a few of the more complicated ones only estimates have been obtained. In [3] the technique for showing a module is infinitely generated is to map it onto a simpler vector space. For the examples studied there, the vector space can be explicitly described and it is obviously infinite dimensional. The purpose of this paper is to introduce a new technique for estimating the dimension of the vector space in more general cases.

At its heart is a surprising discovery linking skein theory and group characters. For a compact orientable 3-manifold M, we show that the character of every representation of $\pi_1(M)$ into $SL_2(\mathbb{C})$ induces a linear functional on the vector space associated to M. The structure of this vector space as an algebra provides a somewhat technical criterion for the existence of an infinite dimensional subspace. However, by invoking a theorem of Culler and Shalen [4] on the structure of the set of $SL_2(\mathbb{C})$ characters, we obtain a much simpler condition.

Let M be a 3-manifold. The Kauffman bracket skein module of M is an algebraic invariant, $K(M)$, built from the set of all framed links in M. By a framed link we mean an embedded collection of annuli considered up to isotopy in M. The set of framed links is denoted $\mathcal{L}_M$ and it includes the empty link $\emptyset$. Three links L, L_0 and L_∞ are said to be *Kauffman skein related* if they can be embedded identically except in a ball where they appear as shown in Figure 1 (assuming, also, that the framing annuli lie flat in the plane of projection). The notation $L \amalg \bigcirc$ indicates the union of L with an unlinked 0-framed unknot.

Let R denote the ring of Laurent polynomials $\mathbb{Z}[A, A^{-1}]$ and $R\mathcal{L}_M$ the free R-module with basis $\mathcal{L}_M$. If L, L_0 and L_∞ are Kauffman skein related then $L - AL_0 - A^{-1}L_\infty$ is called a *skein relation.* For any L in $\mathcal{L}_M$ we say the expression

Received by the editors December 1, 1995.
1991 *Mathematics Subject Classification.* Primary 57M99.
Key words and phrases. Knot, link, 3-manifold, skein module, group characters.

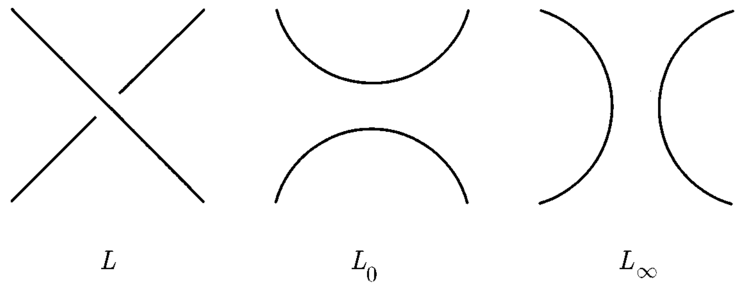

FIGURE 1

$L \amalg \bigcirc + (A^2 + A^{-2})L$ is a *weight relation*.[1] Let $S(M)$ be the smallest submodule of $R\mathcal{L}_M$ containing all possible skein and framing relations. We define $K(M)$ to be the quotient $R\mathcal{L}_M/S(M)$.

We are going to focus on a complex vector space that is closely related to this module but much easier to study. Roughly speaking the vector space is obtained by setting $A = -1$ in $K(M)$ but this is not quite the exact definition. Specifically, let $\mathcal{H}_M$ denote the set of (unframed) links in M, including $\emptyset$, but considered only up to homotopy. Let $\mathbb{C}\mathcal{H}_M$ be the complex vector space with basis $\mathcal{H}_M$. There is a subspace $W(M)$ generated by all the specialized skein relations, $L + L_0 + L_\infty$, along with the single weight relation $\bigcirc + 2\,\emptyset$. The principal object of our investigations will be the vector space $V(M) = \mathbb{C}\mathcal{H}_M/W(M)$.

In the module $K(M)$, specialized at $A = -1$, the skein relations imply that crossings can be changed at will. Furthermore, every weight relation except $\bigcirc + 2\,\emptyset$ is trivial. This is because they are equivalent to the framing relations used in earlier definitions of $K(M)$ [8, 5], which are easily seen to be trivial if $A = -1$. The fact that framing ceases to be relevant seems to make $A = -1$ a more natural specialization than $A = 1$. However, Barrett [1] has shown, for any $t \in \mathbb{C}$, that a spin structure on M induces an isomorphism between specializations at $A = t$ and $A = -t$.

Since framings and crossings are irrelevant in both $V(M)$ and the specialized $K(M)$, the only difference between them is the use of complex coefficients. Clearly a finite set of generators for $K(M)$ would span $V(M)$. It should also be noted that $\mathbb{C}\mathcal{H}_M$ is a commutative algebra with unit $\emptyset$ and multiplication given by disjoint union of links. Furthermore, $W(M)$ is an algebra ideal so the multiplication descends to $V(M)$.

2. RESULTS AND PROOFS

By a representation we mean a homomorphism of groups $\rho : \pi_1(M) \to SL_2(\mathbb{C})$. Its character, χ_ρ, is ρ followed by the trace function. Let $K \in \mathcal{H}_M$ be a knot. After a choice of orientation K can be thought of as a conjugacy class in $\pi_1(M)$. Since the trace of a matrix in $SL_2(\mathbb{C})$ is invariant under conjugation and inversion, it makes sense to speak of $\chi_\rho(K)$ regardless of the choice of orientation. Hence, there is a well defined function on the knots in $\mathcal{H}_M$ given by $K \mapsto -\chi_\rho(K)$. It extends to a function $f_\rho : \mathbb{C}\mathcal{H}_M \to \mathbb{C}$ by requiring it to be a map of algebras.

[1] In Kauffman's original paper [7] on the the bracket polynomial, each trivial circle in a state contributed $-A^2 - A^{-2}$ to its weight.

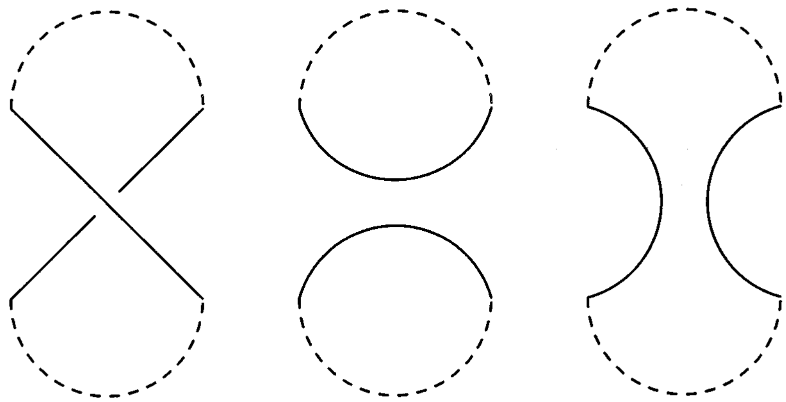

FIGURE 2

Theorem 1. *The map f_ρ descends to a well defined algebra map on $V(M)$.*

Proof. Clearly $f_\rho(\bigcirc + 2\,\emptyset) = 0$. Thus it suffices to show that $f_\rho(L + L_0 + L_\infty) = 0$ for every Kauffman skein triple. By looking at the crossing ball from different directions we can reduce all cases to the one shown in Figure 2, where the dashed arcs indicate how the tangles in the crossing ball are connected. Since f_ρ is an algebra map, we may also assume that L and L_∞ are knots. Let K_1 and K_2 be the components of L_0. With a suitable choice of base point there are elements a and b in $\pi_1(M)$ so that $L \simeq ab$, $L_\infty \simeq ab^{-1}$, $K_1 \simeq a$ and $K_2 \simeq b$. Letting $X = \rho(a)$ and $Y = \rho(b)$ we have

$$f_\rho(L + L_0 + L_\infty) = -\operatorname{tr}(XY) + \operatorname{tr}(X)\operatorname{tr}(Y) - \operatorname{tr}(XY^{-1}).$$

It is an elementary computation to verify that the right-hand side is identically zero in $SL_2(\mathbb{C})$. □

Since f_ρ is an algebra map we can describe its value on certain knots with a sequence of polynomials. Specifically, let $\{p_i \mid i \in \mathbb{N}\} \subset \mathbb{C}[z]$ be the set of polynomials defined by the following recursion:

$$\begin{aligned} p_0 &= -2, \\ p_1 &= -z, \quad \text{and} \\ p_{i+2} &= z p_{i+1} - p_i. \end{aligned}$$

Lemma 1. *Choose $\gamma \in \pi_1(M)$. For each natural number i, let K_i be a knot that is freely homotopic to either γ^i or γ^{-i}. For every representation ρ we have $f_\rho(K_i) = p_i(\chi_\rho(\gamma))$.*

Proof. Up to homotopy, K_0 is $\bigcirc$ and K_i is an $(i,1)$-cable of K_1. Within a regular neighborhood of K_1 there is a diagram of K_{i+2} much like the schematic in Figure 3. The innermost crossing resolves via a skein relation giving $K_{i+2} = -(K_1 K_{i+1}) - K_i$.

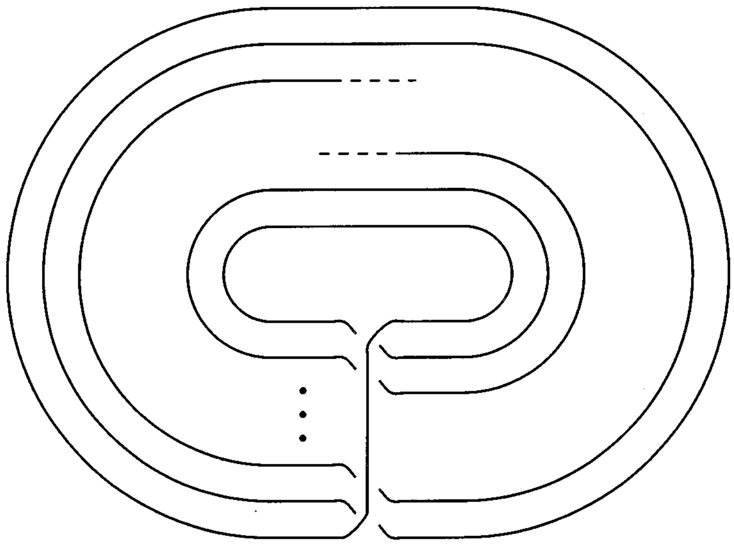

FIGURE 3

Setting $z = \chi_\rho(\gamma)$ we have the following equations:

$$f_\rho(K_0) = -2,$$
$$f_\rho(K_1) = -z, \quad \text{and}$$
$$f_\rho(K_{i+2}) = z f_\rho(K_{i+1}) - f_\rho(K_i).$$

These are identical to the defining relations for p_i. □

It should be fairly obvious from the definition that $\deg p_i = i$. Let M_n denote the $n \times n$ matrix over $\mathbb{C}[z_1, \dots, z_n]$ whose (i, j)-th entry is $p_i(z_j)$.

Lemma 2. *The degree of $|M_n|$ is $n(n+1)/2$.*

Proof. The proof is by induction on n. Clearly it is true for $n = 1$. For larger values we expand the determinant along the n-th row into

$$|M_n| = \sum_{j=1}^{n} p_n(z_j)|C_j|,$$

where each C_j is just M_{n-1} with a different set of indeterminates. By the inductive hypothesis $\deg |C_j| = (n-1)n/2$, so each term in the summation has degree $n(n+1)/2$. The highest degree term of the j-th summand contains z_j^n, but no other indeterminate with exponent larger than $n - 1$. Therefore the degree $n(n+1)/2$ terms cannot cancel. □

From now on, let us denote the set of all characters by $X(M)$. For each $\gamma \in \pi_1(M)$ there is a function $t_\gamma : X(M) \to \mathbb{C}$ given by $\chi_\rho \mapsto \chi_\rho(\gamma)$.

Theorem 2. *If there exists t_γ whose image is open then $V(M)$ is infinite dimensional. In fact, the knots K_i corresponding to powers of γ are linearly independent in $V(M)$.*

Proof. Let $V_n \subset V(M)$ be the span of $\{K_1, \dots, K_n\}$. We will show that $V_n \cong \mathbb{C}^n$ for each n. Let $\mathcal{U} \subset \mathbb{C}$ be the image of the hypothesized t_γ. The zero set of a non-constant polynomial cannot contain an open set, so $|M_n|$ is not identically zero on $\mathcal{U}^n$. Hence we may choose representations $\rho_1, \dots, \rho_n$ so that the matrix $[p_i(\chi_{\rho_j}(\gamma))]$ is nonsingular.

Let $\Phi : V(M) \to \mathbb{C}^n$ be the linear map given by $(f_{\rho_1}, \dots, f_{\rho_n})$. Since $\Phi(V_n)$ is spanned by the rows of $[p_i(\chi_{\rho_j}(\gamma))]$, non-singularity proves that $\Phi|_{V_n}$ is an isomorphism. □

Theorem 2 provides a criterion for determining if $K(M)$ is infinitely generated. A theorem of Culler and Shalen [4] provides an easy test for the existence of t_γ satisfying the hypothesis of Theorem 2.

Theorem 3 (Culler-Shalen)**.** *There exists n such that $X(M)$ is a closed algebraic subset of $\mathbb{C}^n$. In those coordinates, each t_γ is a polynomial map.*

Theorem 4. *If $\dim(X(M)) \geq 1$ then some t_γ has open image.*

Proof. Choose a Zariski component $X_0 \subset X(M)$ whose dimension is at least one. If $t_\gamma|_{X_0}$ were constant for every γ then X_0 would consist of a single point. Hence there is some t_γ that is a non-constant function on X_0. Since X_0 is irreducible t_γ is non-constant on a neighborhood of a manifold point, where it is also holomorphic. The Riemann mapping theorem implies that it has open image. □

Corollary 1. *If $\dim(X(M)) \geq 1$ then $K(M)$ is infinitely generated.*

References

1. J. W. Barrett, *Skein spaces and spin structures,* preprint.
2. D. Bullock, *The $(2,\infty)$-skein module of the complement of a $(2, 2p+1)$ torus knot*, J. Knot Theory Ramifications, **4** no. 4 (1995) 619–632. CMP 96:04
3. ———, *On the Kauffman bracket skein module of surgery on a trefoil*, Pacific J. Math., to appear.
4. M. Culler and P. Shalen, *Varieties of group representations and splittings of 3-manifolds*, Ann. Math., **117** (1983) 109–146. MR **84k:**57005
5. J. Hoste and J. H. Przytycki, *The $(2,\infty)$-skein module of lens spaces; a generalization of the Jones polynomial*, J. Knot Theory Ramifications, **2** no. 3 (1993) 321–333. MR **95b:**57010
6. ———, *The Kauffman bracket skein module of $S^1 \times S^2$*, Math Z., **220** (1995) 65–73. MR **96f:**57006
7. L. Kauffman, *State models and the Jones polynomial*, Topology, **26** no. 3 (1987) 395–401. MR **88f:**57006
8. J. H. Przytycki, *Skein modules of 3-manifolds*, Bull. Pol. Acad. Sci., **39(1-2)** (1991) 91–100. MR **94g:**57011

Department of Mathematics, Boise State University, Boise, Idaho 83725
E-mail address: bullock@math.idbsu.edu