

INVARIANT SUBSPACES FOR POLYNOMIALLY HYPNORMAL OPERATORS

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ABSTRACT. We show that if T is a bounded operator on a Hilbert space such that $p(T)^*p(T) - p(T)p(T)^* \geq 0$ for every polynomial p , then T has a nontrivial invariant subspace.

Let H be a complex separable Hilbert space, and let $L(H)$ denote the algebra of all bounded linear operators on H . An operator $T \in L(H)$ is hyponormal if $T^*T - TT^* \geq 0$ and polynomially hyponormal if $p(T)$ is hyponormal for every analytic polynomial $p \in C[z]$. Recall also that an operator $S \in L(H)$ is subnormal if it is the restriction of a normal operator to an invariant subspace. One can easily show that every subnormal operator is polynomially hyponormal. The question whether the converse is true had been open for a long time but recently, R. Curto and M. Putinar [CuP] answered it negatively.

As we shall see below, polynomially hyponormal operators and subnormal operators still share some nice properties, especially when we look at the lattice of their invariant subspaces. Recall that a subalgebra $A \subset L(H)$ is said to be reflexive if $A = \text{AlgLat}A$, where $\text{AlgLat}A = \{T \in L(H) : \text{Lat}A \subset \text{Lat}T\}$. An operator $T \in L(H)$ is said to be reflexive if the weakly closed subalgebra generated by T in $L(H)$ is reflexive. The main result of this paper is the following theorem, which gives an affirmative answer to a question raised in [CuP, Problem 4.1]:

Theorem 1. *Every polynomially hyponormal operator $T \in L(H)$ is reflexive.*

For the case when T is subnormal, this theorem has been proved by R. Olin and J. Thomson (cf. [OT]). An immediate consequence of Theorem 1 is the following.

Corollary 1. *Every polynomially hyponormal operator has a nontrivial invariant subspace.*

For subnormal operators, the existence of invariant subspaces was proved by S. Brown in [Br1]. Also, a large class of hyponormal operators have invariant subspaces (cf. [Br2]). The proof of Theorem 1 is essentially based on some deep results from the theory of dual algebras. A dual algebra $A \in L(H)$ (i.e. a weak* closed unital subalgebra of $L(H)$) is said to have property $(A_1(r))$ for some $r \geq 1$ if for every weak* continuous linear functional ϕ on A , and for every $s > r$, there exist vectors x and y in H such that $\phi(T) = (Tx, y)$ for every $T \in A$ and moreover, $\|x\| \cdot \|y\| \leq s\|\phi\|$

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(see [BFP, p. 7]). For any $T \in L(H)$ let A_T denote the dual algebra generated by T in $L(H)$. Suppose that $T \in L(H)$ is such that A_T is isometrically isomorphic and weak* homeomorphic with $H^\infty(D)$, the algebra of all bounded analytic functions in the unit disc D . In this case, it was proved independently by H. Bercovici [B] and B. Chevreau [Ch] that A_T has property $(A_1(r))$ for some universal constant $r \geq 1$ (Bercovici gets the best value $r = 1$). Moreover, S. Brown and B. Chevreau proved that every such algebra is reflexive (cf. [BrCh]).

On the other hand, G. Cassier showed that if for a given $T \in L(H)$, the dual algebra A_T is uniform, then it can be written as $A_T = \bigoplus_{i \geq 0} A_i$, where A_0 is an abelian von Neumann algebra and for every $i \geq 1$, A_i is isometrically isomorphic and weak* homeomorphic with $H^\infty(D)$ (cf. [Cas]). A similar decomposition for dual Banach algebras is given in [Pr2]. Recall now that by the von Neumann double commutant theorem, every von Neumann algebra is reflexive. Moreover, if it is abelian, then obviously it also has property $(A_1(1))$. Now, taking direct sums of reflexive algebras with property $(A_1(1))$, we are still in the same class (cf. [BFP, p. 11]). As a consequence of these results, we have the following theorem, that will be very useful in the sequel.

Theorem 2. *Let $T \in L(H)$ be an operator such that A_T is a uniform algebra. Then A_T has property $(A_1(1))$ and is reflexive.*

In order to apply the above theorem, we need one more result.

Proposition 1. *Let $T \in L(H)$ be a polynomially hyponormal operator. Then every operator S in the weakly closed algebra W_T generated by T in $L(H)$ is hyponormal.*

Proof. Let us first observe that the weak and the strong-* closure of the linear space $\{p(T) : p \in C[z]\}$ coincide (see [D, p. 40]). Therefore, given $S \in W_T$, there exists a net $\{p_i(T)\}$ such that $\|p_i(T)x - Sx\| + \|p_i(T)^*x - S^*x\| \rightarrow 0$ for every $x \in H$. Therefore $\|S^*x\| = \lim \|p_i(T)^*x\| \leq \lim \|p_i(T)x\| = \|Sx\|$ for every $x \in H$, hence S is a hyponormal operator. \square

We are now able to prove the following structure theorem for the dual algebra generated by a polynomially hyponormal operator.

Theorem 3. *If $T \in L(H)$ is a polynomially hyponormal operator, then the dual algebra A_T generated by T in $L(H)$ has property $(A_1(1))$ and is reflexive.*

Proof. Let us recall that the spectral radius of a hyponormal operator equals its norm (cf. [Co, p. 47]). Therefore, by Proposition 1, the dual algebra A_T is uniform, and now a simple application of Theorem 2 yields the result. \square

In the case of a subnormal operator, the property $(A_1(r))$ for A_T was proved in [OT]. Subsequently, H. Bercovici and J. Conway showed that $r = 1$ (cf. [BCo]).

Proof of Theorem 1. Follows easily from Theorem 3 and the fact that a dual algebra with property $(A_1(1))$ is also weakly closed (cf. [BFP, p. 11]). \square

We close with a few remarks. Suppose that $T \in L(H)$ is a polynomially hyponormal operator without nonzero normal summand. Then the above mentioned decomposition for uniform dual algebras implies that A_T is isometric and weak* homeomorphic with the algebra of bounded analytic functions on a planar open set G . Moreover, as shown in [Cas], the components of G are simply connected. This shows that polynomially hyponormal operators have a very rich functional calculus.

Suppose now that $T \in L(H)$ is such that $f(T)$ is hyponormal for every rational function f with poles outside $\sigma(T)$. Then one can see that $\sigma(T)$ is a spectral set for T , in other words T is a von Neumann operator. For such operators Theorem 3 has already been proved in [CoDu] and [Pr1]. In this context, the following question seems natural. Is there any polynomially hyponormal operator whose spectrum is not a spectral set? Since any subnormal operator is a von Neumann operator, an affirmative answer to this question would provide another proof of the existence of polynomially hyponormal operators which are not subnormal.

REFERENCES

- [B] H. Bercovici, *Factorization theorems and the structure of operators on Hilbert space*, Annals of Math. **128** (1988), 399–413. MR **89i**:47032
- [BCo] H. Bercovici and J. B. Conway, *A note on the algebra generated by a subnormal operator*, Operator Theory: Adv. Appl. 32, 53–56. MR **90a**:47058
- [BFP] H. Bercovici, C. Foias and C. Pearcy, *Dual Algebras with Applications to Invariant Subspaces and Dilation Theory*, C.B.M.S. Regional Conf. Ser. in Math. no. 56, Amer. Math. Soc., Providence, R.I., 1985. MR **87g**:47091
- [Br1] S. Brown, *Some invariant subspaces for subnormal operators*, Integral Equations and Operator Theory **1** (1978), 310–333. MR **80c**:47007
- [Br2] ———, *Hyponormal operators with thick spectrum have invariant subspaces*, Annals of Math. **125** (1987), 93–103. MR **88c**:47010
- [BrCh] S. Brown and B. Chevreau, *Toute contraction a calcul fonctionnel isometrique est reflexive*, C. R. Acad. Sci. Paris, Serie I **307** (1988), 185–188. MR **89i**:47007
- [Cas] G. Cassier, *Sur la structure d'algebres duales uniformes d'operateurs sur l'espace de Hilbert*, C. R. Acad. Sci. Paris, Serie I **309** (1989), 479–482. MR **91f**:47059
- [Ch] B. Chevreau, *Sur les contractions a calcul fonctionnel isometrique*, II, J. Operator Theory **20** (1988), 269–293. MR **90f**:47021
- [Co] J. B. Conway, *The Theory of Subnormal Operators*, Amer. Math. Soc., Providence, RI, 1991. MR **92h**:47026
- [CoDu] J. B. Conway and J. Dudziak, *Von Neumann operators are reflexive*, J. Reine Angew. Math. **408** (1990), 34–56. MR **91f**:47003
- [CuP] R. Curto and M. Putinar, *Nearly subnormal operators and moment problems*, J. Funct. Analysis **115** (1993), 480–497. MR **95d**:47024
- [D] J. Dixmier, *Les Algebres d'Operateurs dans l'Espace Hilbertien*, Gauthier-Villars, Paris, 1957. MR **20**:1234
- [OT] R. Olin and J. Thomson, *Algebras of subnormal operators*, J. Funct. Analysis **37** (1980), 271–301. MR **82a**:47024
- [Pr1] B. Prunaru, *Von Neumann operators are reflexive*, Integral Equations and Operator Theory **14** (1991), 609–612. MR **92h**:47008
- [Pr2] ———, *A structure theorem for singly generated dual uniform algebras*, Indiana Univ. Math. J. **43** (1994), 729–736. MR **96e**:47048

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