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ON SIMULTANEOUS EXTENSION OF CONTINUOUS PARTIAL FUNCTIONS

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Abstract. For a metric space X let $\mathcal{C}_{vc}(X)$ (that is, the set of all graphs of real-valued continuous functions with a compact domain in X) be equipped with the Hausdorff metric induced by the hyperspace of nonempty closed subsets of $X\times\mathbf{R}$. It is shown that there exists a continuous mapping $\Phi : \mathcal{C}_{vc}(X) \to \mathcal{C}_b(X)$ satisfying the following conditions:

(i) $\Phi(f)|\operatorname{dom} f = f$ for all *partial* functions f.

(ii) For every nonempty compact subset K of X, $\Phi|\mathcal{C}_b(K) : \mathcal{C}_b(K) \to \mathcal{C}_b(X)$ is a linear positive operator such that $\Phi(1_K) = 1_X$.

1. Introduction

Let X be a topological space. Recently V.V. Filippov [4] and his students studied the space $\mathcal{C}_v(X)$ of partial real-valued continuous functions with closed domains in X. (Let us mention that this space originates in [5, 6].):

Consider the set $\mathcal{C}_v(X)$ of all continuous functions $f : A \to \mathbf{R}$, where A denotes a closed subspace of X, and identify each such function with its graph. Equip now $\mathcal{C}_v(X)$ with the topology induced by the Vietoris topology on the hyperspace $\exp(X\times\mathbf{R})$ of all nonempty closed subsets of $X\times\mathbf{R}$. In this paper we shall deal mainly with the subspace $\mathcal{C}_{vc}(X)$ of $\mathcal{C}_v(X)$ that consists of all partial continuous real-valued functions having a compact domain in X. Observe that for a metric space X the space $\mathcal{C}_{vc}(X)$ is metrizable by the Hausdorff metric induced on $X\times\mathbf{R}$ [7].

For an arbitrary topological space we denote by $\exp_c X$ the set of nonempty compact subsets of X equipped with the Vietoris topology. Furthermore let $\Pi : \mathcal{C}_{vc}(X) \to \exp_c X$ be the natural projection. Note that for any $K \in \exp_c X$ the equality $\Pi^{-1}(K) = \mathcal{C}_b(K)$ holds. In this paper $\mathcal{C}_b(X)$ denotes the set of all bounded continuous real-valued functions f on a topological space X endowed with the topology given by the sup-norm $\|f\|_\infty = \sup_{x\in X}|f(x)|$.

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In the fifties J. Dugundji proved the following theorem.

Theorem 1 ([2]). *Let X be a metric space and A a closed subspace of X : Then there exists a linear operator Φ which makes correspond to each $f \in \mathcal{C}_b(A)$ an extension $\Phi(f) \in \mathcal{C}_b(X)$.*

For a discussion of some related positive and negative results we refer the reader to [11].

On the other hand E.N. Stepanova [10] proved recently that for a metric space X it is possible to extend all elements of $\mathcal{C}_{vc}(X)$ simultaneously to X in the following sense.

Theorem 2 ([10]). *For any metrizable space X there exists a continuous mapping $\alpha : \mathcal{C}_{vc}(X) \to \mathcal{C}_b(X)$ such that for any function f, $\alpha(f)|\operatorname{dom} f = f$ and $\sup_{x \in \operatorname{dom} f} |f(x)| = \sup_{x \in X} |\alpha(f)(x)|$.*

She showed moreover that among the Hausdorff paracompact p-spaces only metrizable spaces have such a mapping α.

In this paper we want to prove the following theorem.

Theorem 3. *Let X be a metrizable space. Then there exists a continuous mapping $\Phi : \mathcal{C}_{vc}(X) \to \mathcal{C}_b(X)$ that satisfies the following conditions:*
(i) *$\Phi(f)|\operatorname{dom} f = f$ for each partial function f;*
(ii) *for any $K \in \exp_c X$ the restriction $\Phi|\Pi^{-1}(K)$ is a linear positive operator such that $\Phi(1_K) = 1_X$.*

Observe that for compact domains our result generalizes simultaneously the theorems of Dugundji and Stepanova.

Remark 1. It is readily seen that each one-to-one (continuous) preimage of a space having an extension operator as described in Theorem 3 also admits such an operator. Therefore every one-to-one preimage of a metric space (i.e. a submetrizable space) admits an operator of this kind. Note that the class of one-to-one preimages of metric spaces contains all paracompact σ-spaces (compare [3, Exercise 5.5.7]). We deduce in particular that each stratifiable space admits an extension operator as described in Theorem 3. Observe however that the one-point-Lindelöfication of an uncountable discrete space admits such an operator, although it is not the one-to-one preimage of a metric space.

2. Selections

Our proof of Theorem 3 will be based on the following well-known selection theorem due to E. Michael [8, Theorem 3.2"]. Let $F : X \to Y$ be a lower semicontinuous multi-valued mapping from a paracompact Hausdorff space to a Banach space such that for each $x \in X$, $F(x)$ is a nonempty closed convex subset of Y. Then there exists a continuous mapping $f : X \to Y$ such that $f(x) \in F(x)$ whenever $x \in X$.

We recall that a multi-valued mapping $F : X \to Y$ is called *lower semicontinuous* provided that $\{x \in X : F(x) \cap U \neq \emptyset\}$ is open in X whenever U is open in Y.

For any metric space (X, d), any $\delta > 0$ and $A \subseteq X$ we set $B(A, \delta) = \{y \in X : d(a, y) < \delta \text{ for some } a \in A\}$.

The proof of Theorem 3 consists of the verification of three lemmas.

Lemma 1. *Let K and L be compact subsets of a metric space (X,d), $r : X \to K$ a continuous mapping and ϵ a positive real number such that $\max_{x\in L} d(x, r(x)) < \epsilon$. Then there exist a locally finite open cover $\gamma = \{V_t : t \in T\}$ of $X \setminus L$ and $\delta > 0$ such that for any $t \in T$ with $V_t \cap B(L,\delta) \neq \emptyset$ we can choose $a_t \in K$, $b_t \in L$ and $y_t \in X \setminus L$ satisfying the following three conditions:*
(i) $V_t \subseteq r^{-1}(B(a_t, 2\epsilon)) \cap B(y_t, \frac{1}{4}d(y_t, L))$;
(ii) $d(a_t, b_t) < 2\epsilon$; *and*
(iii) $d(y_t, b_t) \leq \frac{5}{4}d(y_t, L)$.

Proof. Note that $\mathcal{C} := \{B(y, \frac{1}{4}d(y,L)) \cap r^{-1}(B(r(y), 2\epsilon)) : y \in X \setminus L\}$ is an open cover of $X \setminus L$. Since the subspace $X \setminus L$ of X is paracompact, there is a locally finite open refinement $\gamma = \{V_t : t \in T\}$ of $\mathcal{C}$. We shall show that γ satisfies the condition formulated in Lemma 1. Assume the contrary. Then for each $n \in \omega$ there is an element $V_{t_n} \in \gamma$ that hits $B(L, 2^{-n})$, but we cannot find a triple $(a_{t_n}, b_{t_n}, y_{t_n})$ as described in Lemma 1.

Fix $n \in \omega$. There is $y_{t_n} \in X \setminus L$ such that $V_{t_n} \subseteq B(y_{t_n}, \frac{1}{4}d(y_{t_n}, L))$.

Next we want to verify that $d(y_{t_n}, L) < \frac{4}{3} \cdot 2^{-n}$. Indeed, let $y \in V_{t_n} \cap B(L, 2^{-n})$. Thus $d(y_{t_n}, y) < \frac{1}{4}d(y_{t_n}, L)$. Let $z \in L$ be such that $d(y,z) = d(y,L)$. We have $d(y_{t_n}, L) \leq d(y_{t_n}, z) \leq d(y_{t_n}, y) + d(y,z) < \frac{1}{4}d(y_{t_n}, L) + 2^{-n}$.

Therefore $\frac{3}{4}d(y_{t_n}, L) < 2^{-n}$ and thus $d(y_{t_n}, L) < \frac{4}{3} \cdot 2^{-n}$.

It follows that $L \cup \{y_{t_n} : n \in \omega\}$ is a compact subset of X. Without loss of generality we can suppose that $(y_{t_n})_{n\in\omega}$ is a convergent sequence and that $y := \lim_{n\to\infty} y_{t_n} \in L$. For each $n \in \omega$ let $z_{t_n} \in L$ be such that $d(y_{t_n}, z_{t_n}) = d(y_{t_n}, L)$. Then $y = \lim_{n\to\infty} z_{t_n}$, because $d(y_{t_n}, z_{t_n}) \to 0$. Since $d(y, r(y)) < \epsilon$ and r is continuous, there exists $\delta > 0$ such that $\delta < \epsilon$, $B(y,\delta) \subseteq B(r(y), 2\epsilon)$ and $r(B(y,\delta)) \subseteq B(r(y), 2\epsilon)$.

Furthermore there exists $m \in \omega$ such that $B(y_{t_m}, \frac{1}{4}d(y_{t_m}, L)) \subseteq B(y,\delta)$ and $z_{t_m} \in B(y,\delta)$. Consequently for V_{t_m} we can define the triple $(a_{t_m} = r(y), b_{t_m} = z_{t_m}, y_{t_m})$. By our choice $V_{t_m} \subseteq B(y_{t_m}, \frac{1}{4}d(y_{t_m}, L)) \subseteq B(y,\delta) \subseteq r^{-1}(B(r(y), 2\epsilon)) = r^{-1}(B(a_{t_m}, 2\epsilon))$. Moreover $d(a_{t_m}, b_{t_m}) = d(r(y), z_{t_m}) \leq d(r(y), y) + d(y, z_{t_m}) < \epsilon + \delta < 2\epsilon$. Finally $d(y_{t_m}, b_{t_m}) = d(y_{t_m}, z_{t_m}) = d(y_{t_m}, L) \leq \frac{5}{4}d(y_{t_m}, L)$.

Hence the defined triple satisfies conditions (i)–(iii) of Lemma 1. We have reached a contradiction and conclude that the cover γ possesses all the properties described in Lemma 1. ☐

For a compact subset K of a Banach space X, $\overline{\mathrm{co}}K$ will denote its closed convex hull in X. Note that $\overline{\mathrm{co}}K$ is compact, too (e.g. [9, Theorem 3.20]).

Let $\exp_c X$ be the hyperspace of nonempty compact subsets of X equipped with the Hausdorff metric, which we shall denote by dist. Observe that for any sequence $(K_n)_{n\in\omega}$ converging to some point K in $\exp_c X$, we have $\overline{\mathrm{co}}K_n \to \overline{\mathrm{co}}K$. It is known that the Hausdorff metric induces the Vietoris topology on $\exp_c X$ [7].

Lemma 2. *Let X be a Banach space and let $\mathcal{C}_b(X,X)$ be the Banach space of all bounded continuous X-valued functions on X equipped with its* sup*-norm. Then there exists a continuous mapping $\mu : \exp_c X \to \mathcal{C}_b(X,X)$ such that for each $K \in \exp_c X$ we have $\mu(K)(X) \subseteq \overline{\mathrm{co}}K$ and $\mu(K)|K = \mathrm{id}_K$.*

Proof. Let $R : \exp_c X \to \mathcal{C}_b(X,X)$ be the multi-valued mapping defined by $K \mapsto \{r \in \mathcal{C}_b(X,X) : r(X) \subseteq \overline{\mathrm{co}}K \text{ and } r|K = \mathrm{id}_K\}$.

Note that for each $K \in \exp_c X$, $R(K)$ is a closed convex subset of $\mathcal{C}_b(X,X)$. It is nonempty by Dugundji's theorem [2, Theorem 4.1].

Since $\exp_c X$ is metrizable, it remains to be shown that R is lower semi-continuous; then we can apply Michael's selection theorem in order to obtain the continuous mapping μ described in Lemma 2.

Let U be an open subset of $\mathcal{C}_b(X,X)$ and let $K \in \exp_c X$ be such that $R(K) \cap U \neq \emptyset$. If K is not an interior point of $\{K \in \exp_c X : R(K) \cap U \neq \emptyset\}$, then there exists a sequence $(K_n)_{n\in\omega}$ in $\exp_c X$ converging to K such that $R(K_n) \cap U = \emptyset$ whenever $n \in \omega$. Fix $r \in R(K) \cap U$ and let $\epsilon > 0$ be such that $B(r, 2\epsilon) \subseteq U$. For each $n \in \omega$ set $d_n = \max_{x\in K_n} \|r(x) - x\|$.

Since r is continuous, $K_n \to K$ and $r|K = \mathrm{id}_K$, it is readily seen that the sequence $(d_n)_{n\in\omega}$ converges to 0. Set $\delta = \frac{\epsilon}{4}$ and find $N \in \omega$ such that for any $n \in \omega$ with $n > N$ we have $\max\{\mathrm{dist}(\overline{\mathrm{co}}K, \overline{\mathrm{co}}K_n), d_n\} < \delta$.

Fix $n \in \omega$ such that $n > N$. We want to define $r_n \in R(K_n)$ such that

$$\sup_{x\in X} \|r(x) - r_n(x)\| \leq \epsilon.$$

To this end we are going to apply Lemma 1. The role of L is now played by K_n, the role of K is played by $\overline{\mathrm{co}}K$ and the role of ϵ is played by δ.

By Lemma 1 there exist a locally finite open cover $\gamma = \{V_t : t \in T\}$ of $X \setminus K_n$ and a neighborhood V of K_n in X such that for each V_t that hits V we can find a triple $(a_t, b_t, y_t) \in \overline{\mathrm{co}}K \times K_n \times (X \setminus K_n)$ satisfying

(i) $V_t \subseteq r^{-1}(B(a_t, 2\delta)) \cap B(y_t, \frac{1}{4}\|y_t - K_n\|)$;

(ii) $\|a_t - b_t\| < 2\delta$; and

(iii) $\|y_t - b_t\| < \frac{5}{4}\|y_t - K_n\|$.

If $V_t \cap V = \emptyset$, then in the light of the definition of the cover $\mathcal{C}$ in the proof of Lemma 1 we can choose $a_t \in \overline{\mathrm{co}}K$ such that $V_t \subseteq r^{-1}(B(a_t, 2\delta))$. Furthermore, since $\mathrm{dist}(\overline{\mathrm{co}}K, \overline{\mathrm{co}}K_n) < \delta$ we find $b_t \in \overline{\mathrm{co}}K_n$ such that $\|a_t - b_t\| < 2\delta$.

Now let $\{g_t : t \in T\}$ be a partition of unity on $X \setminus K_n$ subordinated to $\{V_t : t \in T\}$. Define $r_n : X \to X$ by $r_n(x) = x$ if $x \in K_n$ and $r_n(x) = \Sigma_{t\in T} g_t(x) b_t$ if $x \in X \setminus K_n$.

Note that $r_n(X) \subseteq \overline{\mathrm{co}}K_n$ and $r_n|K_n = \mathrm{id}_{K_n}$. Clearly r_n is continuous at any $x \in X \setminus K_n$. Since inside $V \supseteq K_n$ our construction coincides with the one used by Dugundji as outlined by R. Engelking in [3, Exercise 4.5.20], we see that r_n is also continuous at any $x \in K_n$.

Let us estimate $\sup_{x\in X} \|r(x) - r_n(x)\|$.

Case 1. Suppose that $x \in K_n$. Then $\|r(x) - r_n(x)\| = \|r(x) - x\| \leq d_n < \delta = \frac{\epsilon}{4}$.

Case 2. Suppose that $x \in X \setminus K_n$ and that $\{V_{t_i} : i = 1, \ldots, m\}$ is the collection of all the members $V_t \in \gamma$ such that $x \in V_t$. Then $r_n(x) = \Sigma_{i=1}^m g_{t_i}(x) b_{t_i}$. Since $r(\bigcap_{i=1}^m V_{t_i}) \subseteq \bigcap_{i=1}^m B(a_{t_i}, 2\delta)$, we have $\|r(x) - a_{t_i}\| < 2\delta$ whenever $i \leq m$.

Thus

$$\begin{aligned} \|r(x) - r_n(x)\| &= \|r(x) - \Sigma_{i=1}^m g_{t_i}(x) b_{t_i}\| \\ &\leq \|r(x) - \Sigma_{i=1}^m g_{t_i}(x) a_{t_i}\| + \|\Sigma_{i=1}^m g_{t_i}(x) a_{t_i} - \Sigma_{i=1}^m g_{t_i}(x) b_{t_i}\| \\ &= \|\Sigma_{i=1}^m g_{t_i}(x) r(x) - \Sigma_{i=1}^m g_{t_i}(x) a_{t_i}\| + \|\Sigma_{i=1}^m g_{t_i}(x) a_{t_i} - \Sigma_{i=1}^m g_{t_i}(x) b_{t_i}\| \\ &\leq (\Sigma_{i=1}^m g_{t_i}(x))\|r(x) - a_{t_i}\| + (\Sigma_{i=1}^m g_{t_i}(x))\|a_{t_i} - b_{t_i}\| < 2\delta + 2\delta = \epsilon. \end{aligned}$$

We conclude that $\sup_{x\in X} \|r(x) - r_n(x)\| \leq \epsilon$. It follows that $r_n \in R(K_n) \cap U$. We have reached a contradiction and deduce that $R : \exp_c X \to \mathcal{C}_b(X,X)$ is lower semi-continuous. □

3. Measures

Before proving Lemma 3 let us recall some facts established in [1]. Let (X,d) be a metric space. Denote the σ-algebra of Borel sets of X by $\mathcal{B}(X)$ and let $\mathcal{M}(X)$ be the set of all finite, real-valued, countably additive functions defined on $\mathcal{B}(X)$.

A bounded real-valued function f on X is called *Lipschitzian* if

$$\|f\|_L = \sup\{\frac{|f(x)-f(y)|}{d(x,y)} : d(x,y) \neq 0\} < \infty.$$

On the set $BL(X,d)$ of all such functions a norm is defined by setting $\|f\|_{BL} = \|f\|_\infty + \|f\|_L$. Then $(BL(X,d), \|\cdot\|_{BL})$ is a Banach algebra according to [1].

We shall denote its dual space by $BL(X,d)^*$ and the corresponding norm by $\|T\|_{BL}^* = \sup\{|T(f)| : \|f\|_{BL} = 1\}$. By $\mu \in \mathcal{M}(X) \mapsto \mu(f) = \int f d\mu \in BL(X,d)^*$ an injection of $\mathcal{M}(X)$ into $BL(X,d)^*$ is defined.

In this way we can consider $\mathcal{M}(X)$ a subset of $BL(X,d)^*$. The topology induced on $\mathcal{M}(X)$ by the norm on $BL(X,d)^*$ will be denoted by TBL^*. The weak* topology TW^* on $\mathcal{M}(X)$ is determined by $\mu_n \to \mu$ iff $\int f d\mu_n \to \int f d\mu$ whenever f in $\mathcal{C}_b(X)$. It coincides with TBL^* on the set $\mathcal{M}_s^+(X)$ (of nonnegative measures with separable supports) [1, Theorem 18].

Let us denote by $\mathcal{P}_c(X)$ the space of the probability measures with compact supports on X. Then $\mathcal{P}_c(X) \subseteq \mathcal{M}_s^+(X)$ and the weak* topology on $\mathcal{P}_c(X)$ coincides with the topology induced on $\mathcal{P}_c(X)$ by $\|\cdot\|_{BL}^*$.

It is known that $x \mapsto \delta_x$ defines a topological embedding of X into $BL(X,d)^*$. Here, as usual, δ_x denotes the Dirac measure.

Lemma 3. *Let* $\mu : \exp_c X \to \mathcal{C}_b(X, BL(X,d)^*)$ *be a continuous mapping such that for each* $K \in \exp_c X$ *we have that* $\mu(K)(X) \subseteq \mathcal{P}(K)$ *(where* $\mathcal{P}(K)$ *denotes the space of probability measures on* K*) and* $\mu(K)|K = \mathrm{id}_K$ *(modulo the identification* $x \mapsto \delta_x$*).*

Then the mapping $\Phi : \mathcal{C}_{vc}(X) \to \mathcal{C}_b(X)$ *given by the formula*

$$\Phi(f)(x) = \int f d\mu(\operatorname{dom} f)(x)$$

for every partial function f *and every* $x \in X$ *satisfies the conditions of Theorem* 3.

Remark 2. Let us note that Theorem 3 is an immediate consequence of Lemma 2 (applied to the Banach space $BL(X,d)^*$) and Lemma 3, because $\mathcal{P}(K)$ is a closed convex set in $BL(X,d)^*$.

Proof of Lemma 3. Note first that $\Phi(f)$ is an extension of f, since for any $x \in \operatorname{dom} f$ we have $\mu(\operatorname{dom} f)(x) = \delta_x$ and therefore $\Phi(f)(x) = \int f d\delta_x = f(x)$.

Clearly for any $K \in \exp_c X, \Phi : \mathcal{C}_b(K) \to \mathcal{C}_b(X)$ is a linear positive operator such that $\sup_{x\in X} |\Phi(f)(x)| = \sup_{x \in \operatorname{dom} f} |f(x)|$ and $\Phi(1_K) = 1_X$. Furthermore $\Phi(f)$ is continuous, since $\mathcal{P}(\operatorname{dom} f)$ is endowed with the weak* topology.

It remains to check the continuity of Φ. To this end suppose that $f_n \to f$ in $\mathcal{C}_{vc}(X)$, $K = \operatorname{dom} f$ and $K_n = \operatorname{dom} f_n$ whenever $n \in \omega$. In the following the Hausdorff metric on $\mathcal{C}_{vc}(X)$ will be denoted by dist.

Obviously $K_n \to K$ in $\exp_c X$, since $f_n \to f$. Therefore $\widetilde{K} = K \cup \bigcup\{K_n : n \in \omega\}$ is compact in X [7, Theorem 2.5]. Let us fix an arbitrary continuous real-valued extension $\widetilde{f}$ of f to X. We can consider $\widetilde{f}$ a continuous real-valued function defined on $\mathcal{P}(\widetilde{K})$ by setting $\widetilde{f}(\mu) = \int \widetilde{f} d\mu$ whenever $\mu \in \mathcal{P}(\widetilde{K})$. Note that $\widetilde{f}$ is uniformly continuous with respect to the norm $\|\cdot\|_{BL}^*$ restricted to $\mathcal{P}(\widetilde{K})$.

Define the partial continuous functions $g_n : K_n \to \mathbf{R}$ by setting $g_n = \widetilde{f}|K_n$ where $n \in \omega$. It follows that $g_n \to f$ in $\mathcal{C}_{vc}(X)$, because $K_n \to K$ (in $\exp_c X$). Then $\operatorname{dist}(f_n, g_n) \to 0$ in the metric space $\mathcal{C}_{vc}(X)$. This implies that

$$\max_{x\in K_n} |f_n(x) - g_n(x)| = \max_{x\in K_n} |f_n(x) - \widetilde{f}(x)| \to 0,$$

as it is readily verified.

Now let $\epsilon > 0$. We want to find $N \in \omega$ such that for all $n \in \omega$ with $n > N$ and all $x \in X$, $|\Phi(f)(x) - \Phi(f_n)(x)| < \epsilon$.

By uniform continuity of $\widetilde{f}$ on $\mathcal{P}(\widetilde{K})$ we first find $\delta > 0$ such that if $\mu, \nu \in \mathcal{P}(\widetilde{K})$ and $\|\mu - \nu\|^*_{BL} < \delta$, then $|\int \widetilde{f} d\mu - \int \widetilde{f} d\nu| < \frac{\epsilon}{2}$.

Furthermore there exists $N_1 \in \omega$ such that for all $n \in \omega$ with $n > N_1$ we have $\max_{x\in K_n} |f_n(x) - \widetilde{f}(x)| < \frac{\epsilon}{2}$. Moreover there exists $N_2 \in \omega$ such that for all $n \in \omega$ with $n > N_2$ and all $x \in X$ we have that $\|\mu(K)(x) - \mu(K_n)(x)\|^*_{BL} < \delta$, because μ is continuous.

Let $N = \max\{N_1, N_2\}$. For all $n \in \omega$ such that $n > N$ and all $x \in X$, we obtain finally:

$$|\Phi(f)(x) - \Phi(f_n)(x)| = |\int f d\mu(K)(x) - \int f_n d\mu(K_n)(x)|$$

$$\leq |\int f d\mu(K)(x) - \int \widetilde{f} d\mu(K_n)(x)| + |\int \widetilde{f} d\mu(K_n)(x) - \int f_n d\mu(K_n)(x)|$$

$$= |\int \widetilde{f} d\mu(K)(x) - \int \widetilde{f} d\mu(K_n)(x)| + |\int \widetilde{f} d\mu(K_n)(x) - \int f_n d\mu(K_n)(x)|$$

$$< \frac{\epsilon}{2} + \max_{x\in K_n} |\widetilde{f}(x) - f_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We have shown that Φ is continuous. □

Remark 3. Let us finally remark that it is possible to obtain a variant of Theorem 3 by replacing the reals by an arbitrary Banach space over the reals. In fact we can prove the following result.

If E is a Hausdorff sequentially complete locally convex topological vector space over the reals, then there exists a continuous mapping $\Phi : \mathcal{C}_{vc}(X, E) \to C(X, E)$ (where $C(X, E)$ is equipped with the topology of uniform convergence) that satisfies the following conditions:

(i) $\Phi(f)|\operatorname{dom} f = f$ *for any partial function* $f \in C_{vc}(X, E)$*;*

(ii) $\operatorname{range}(\Phi(f)) \subseteq \overline{\operatorname{co}}(\operatorname{range} f)$*;*

(iii) *for any* $K \in \exp_c X$ *the restriction* $\Phi|\Pi^{-1}(K)$ *is a linear operator from* $C(K, E)$ *into* $C(X, E)$*.*

Indeed, according to Lemmas 2 and 3 let $\mu : \exp_c X \to \mathcal{C}_b(X, BL(X, d)^*)$ be a continuous mapping such that for each $K \in \exp_c X$ we have that $\mu(K)(X) \subseteq \mathcal{P}(K)$ (where $\mathcal{P}(K)$ denotes the space of probability measures on K) and $\mu(K)|K = \mathrm{id}_K$ (modulo the identification $x \mapsto \delta_x$).

First we assume that E is a Banach space. Then the formula $\Phi(f)(x) = \int f\, d\mu(\operatorname{dom} f)(x)$ gives the desired mapping. In order to check the statement observe that $\Phi(f)$ is a continuous mapping from X into E: Since if $x_n \to x$ in X, then $\int f\, d\mu(\operatorname{dom} f)(x_n)$ weakly converges to $\int f\, d\mu(\operatorname{dom} f)(x)$; but

$$\int f\, d\mu(\operatorname{dom} f)(x_n) \in \overline{\operatorname{co}}(\operatorname{range}\ f)$$

and $\overline{\mathrm{co}}(\text{range } f)$ is a compact space. Therefore weak convergence coincides with the usual convergence in E. The proof of the continuity of Φ is the same as in the proof of Lemma 3 (instead of $\mathbf{R}$ we are working with E).

Now suppose that E is a Hausdorff sequentially complete LCTVS. Then we can assume that E is a subspace of a product of Banach spaces $\prod\{E_\alpha : \alpha \in A\}$.

Hence we set $\Phi(f)(x) := \int f \; d\mu(\operatorname{dom} f)(x) = \{\int \mathrm{pr}_\alpha \circ f \; d\mu(\operatorname{dom} f)(x) : \alpha \in A\}$. Let us note that according to sequential completeness of E, $\int f \, d\mu(\operatorname{dom} f)(x)$ belongs to E : Clearly $\mu(\operatorname{dom} f)(x)$ is a weak limit of a sequence $\{\mu_n : n \in \omega\}$ where $\operatorname{supp} \mu_n \subseteq \operatorname{dom} f$ and $|\operatorname{supp} \mu_n| < \omega$. Consequently $\int f \, d\mu_n \in \mathrm{co}(\text{range} f)$ and $\{\int f \, d\mu_n : n \in \omega\}$ is a Cauchy sequence in E. Thus $\int f \, d\mu(\operatorname{dom} f)(x) \in E$ as a limit of a Cauchy sequence.

The continuity of Φ follows from the continuity of the mappings Φ_α, where $(\Phi_\alpha(f))(x) = \int \mathrm{pr}_\alpha \circ f \, d\mu(\operatorname{dom} f)(x)$.

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