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COMPLEX SPECIALIZATIONS OF THE REDUCED GASSNER REPRESENTATION OF THE PURE BRAID GROUP

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Abstract. We will give a necessary and sufficient condition for the specialization of the reduced Gassner representation $G_n(z) : P_n \to GL_{n-1}(\mathbb{C})$ to be irreducible. It will be shown that for $z = (z_1, \dots , z_n) \in (\mathbb{C}^*)^n$, $G_n(z)$ is irreducible if and only if $z_1 \dots z_n \neq 1$.

Introduction

In this paper, we will give a necessary and sufficient condition for the complex specialization of the reduced Gassner representation of the pure braid group, P_n, to be irreducible. The argument that will be presented here is purely algebraic.

In section 1, we will state a theorem that gives a classification, up to equivalence, of all irreducible complex representations of a free group of a certain rank, for which the image of each generator is a *pseudoreflection*. Then, as an application, we state a necessary and sufficient condition for the irreducibility of the complex specialization of the Burau representation of the braid group, B_n. This was proved in [4, p.9]. In section 2, we will define, up to equivalence, the Gassner representation of P_n and show that it is reducible to an $n-1$ representation called the reduced Gassner representation which turns out to be irreducible in $GL_{n-1}(\mathbb{Z}[y_1^{\pm 1}, \dots , y_n^{\pm 1}])$. To apply Theorem 2, which is stated in section 1, for the complex specialization of the reduced Gassner representation of P_n, we construct a certain normal free subgroup of P_n of rank $n-1$; this will be done in section 3 and will be used as a tool to prove our main theorem in section 4.

Main Theorem. *Let $G_n(z) : P_n \to GL_{n-1}(\mathbb{C})$ be the reduced Gassner representation obtained by the specialization: $y_i \to z_i$, where $z = (z_1, \dots , z_n) \in (\mathbb{C}^*)^n$ and the y_i's are the independent indeterminates used in defining this representation. Then $G_n(z)$ is irreducible if and only if $z_1 \dots z_n \neq 1$.*

1. Complex specializations of the reduced Burau representation

Definition 1. Let $\mathbb{C}^r = r \times 1$ (or column) vectors, $\overline{\mathbb{C}}^r = 1 \times r$ (or row) vectors. A matrix $X \in M_r(\mathbb{C})$ is a pseudoreflection if $X - I$ has rank 1. We regard $M_r(\mathbb{C})$ as

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acting from the left on column vectors so that eigenvectors and invariant subspaces lie in $\mathbb{C}^r$. If X is a pseudoreflection, then

$$X = I - AB,$$

where $A \in \mathbb{C}^r$ and $B \in \overline{\mathbb{C}^r}$. This representation is unique, except that if $a \in \mathbb{C}^*$, then $X = I - AB = I - (aA)(a^{-1}B)$. The eigenvalues of X are 1, with multiplicity $r-1$, and $1 - BA$ with multiplicity 1. Hence X is invertible if and only if $BA \neq 1$.

We say G is an irreducible subgroup of $GL_r(\mathbb{C})$ if it spans $GL_r(\mathbb{C})$ as a vectorspace over $\mathbb{C}$. To know the connection between "irreducible subgroup" and "irreducible representation of a group", see [3, p.167].

The following theorem gives a criterion for the group generated by r pseudoreflections in $GL_r(\mathbb{C})$ to be irreducible. We will use this result to find conditions for the Burau and Gassner representations in $GL_n(\mathbb{C})$ to be irreducible.

Theorem 2 ([4, p.6]). *Let $X_1 = I - A_1B_1, \dots, X_r = I - A_rB_r$ be r invertible* pseudoreflections *in $M_r(\mathbb{C})$, where $r \geq 2$. Let τ be the directed graph whose vertices are $1, 2, \dots, r$, and which has a directed edge from i to j ($i \neq j$) precisely when $B_iA_j \neq 0$. Let G be the subgroup of $GL_r(\mathbb{C})$ generated by $X_1, \dots, X_r$. Then*

(a) *The following are equivalent.*

1. *G is an irreducible subgroup of $GL_r(\mathbb{C})$.*
2. *For each $i \neq j$, $1 \leq i, j \leq r$, the graph τ contains a directed path from i to j, $\{A_1, \dots, A_r\}$ is a basis for $\mathbb{C}^r$, and $\{B_1, \dots, B_r\}$ is a basis for $\overline{\mathbb{C}}^r$.*
3. *For each $i \neq j$, $1 \leq i, j \leq r$, the graph τ contains a directed path from i to j, and $(B_iA_j) \in M_r(\mathbb{C})$ is invertible. ((B_iA_j) is the matrix whose entries are B_iA_j for different values of i and j.)*

(b) *Suppose that $G = \langle X_1, \dots, X_r \rangle$ and $H = \langle Y_1, \dots, Y_r \rangle$ are irreducible subgroups of $GL_r(\mathbb{C})$ generated by pseudoreflections $X_i = I - A_iB_i$ and $Y_i = I - C_iD_i$. Then there is a matrix $T \in GL_r(\mathbb{C})$ such that $TX_iT^{-1} = Y_i$ for $i = 1, \dots, r \iff$ there exist $a_1, \dots, a_r \in \mathbb{C}^*$ such that $D_iC_j = a_i^{-1}a_jB_iA_j$, i.e. (B_iA_j) and (D_iC_j) are congugate by a diagonal matrix.*

(c) *If $G = \langle X_1, \dots, X_r \rangle$ is an irreducible subgroup of $GL_r(\mathbb{C})$, generated by pseudoreflections $X_1, \dots, X_r$, then there is a matrix $T \in GL_r(\mathbb{C})$ such that $TX_iT^{-1} = I - P_iQ_i$ where $Q_i = (0, 0, \dots, 1, 0, \dots, 0)$, the i–th standard basis vector of $\overline{\mathbb{C}}^r$.*

Part (b) of Theorem 2 is a classification, up to equivalence, of all irreducible representations

$$F_r\langle x_1, \dots, x_r \rangle \to GL_r(\mathbb{C}),$$

where $F_r\langle x_1, \dots, x_r \rangle$ is a free group of rank r, for which the image of each x_i is a pseudoreflection. Although this is a rather restricted class of representations, it includes the reduced Burau representation and the restriction of the reduced Gassner representation of the pure braid group to a certain normal free subgroup, the normal closure of the square of one of the standard generators of the braid group. We will now show that the images of the generators of B_n, namely, $\sigma_1, \dots, \sigma_{n-1}$, under the complex specialization of the reduced Burau representation are pseudoreflections. Here B_n denotes the braid group on n strings. We will deal with B_n as an abstract group with generators and relations. For more details, see [5, p.7] and [2, p.118]. Then we will apply Theorem 2 to specializations of the reduced Burau representation

$$\overline{\beta_n}(z) : B_n \to GL_{n-1}(\mathbb{C}).$$

Then $\overline{\beta_n}(z) = I - A_i B_i$, where

$$A_1 = (z+1 \quad 1 \quad 0\ldots \quad 0)^T,$$

$$A_i = (\underbrace{0\ldots0}_{i-2} \quad z \quad z+1 \quad 1 \quad \underbrace{0\ldots0}_{n-i-2})^T,$$

$i = 2, \ldots, n-2$, and

$$A_{n-1} = (0 \quad 0\ldots \quad 0 \quad z \quad z+1)^T,$$

where T is the transpose, and $\{B_1, \ldots, B_{n-1}\}$ is the standard basis of $\overline{\mathbb{C}}^{n-1}$. The associated matrix $(B_i A_j)$ is

$$\begin{pmatrix} z+1 & z & 0 & \ldots & 0 & 0 \\ 1 & z+1 & z & \ldots & 0 & 0 \\ 0 & 1 & z+1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & z+1 & z \\ 0 & 0 & 0 & \ldots & 1 & z+1 \end{pmatrix}.$$

Lemma 3 ([4, p.9]). *For $z \in \mathbb{C}^*$, $\overline{\beta_n}(z) : B_n \to GL_{n-1}(\mathbb{C})$ is irreducible if and only if z is not a root of $f_n(t) = t^{n-1} + t^{n-2} + \ldots + t + 1$.*

Proof. A simple induction shows that $det(B_i A_j) = z^{n-1} + z^{n-2} + \ldots + z + 1$. Then by Theorem 2, the result follows. □

2. The Gassner representation of P_n

The pure braid group, P_n, is defined as the kernel of the homomorphism $B_n \to S_n$ defined by $\sigma_i \to (i, i+1)$, $\quad 1 \le i \le n-1$. It admits a presentation with generators

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}; \quad 1 \le i < j \le n.$$

These generators have relations among them. For more details, see [5, pp.19-21]. The Gassner representation of P_n , denoted by γ_n, is defined explicitly in [1, p.15] and [2, p.119]. Then it is clear that the image of ${A_{ij}}^{-1}$ under γ_n is a pseudoreflection X_{ij} where $X_{ij} = I - P_{ij}Q_{ij}$, and P_{ij} is the column vector defined by

$$(\underbrace{0\ldots0}_{i-1} \quad y_j - 1 \quad (y_i-1)(1-y_j)\ldots \quad (y_i-1)(1-y_j) \quad (y_i-1)(-y_j) \quad \underbrace{0\ldots0}_{n-j})^T$$

and

$$Q_{ij} = (\underbrace{0\ldots0}_{i-1} \quad -1 \quad 0\ldots \quad 0 \quad 1 \quad \underbrace{0\ldots0}_{n-j}).$$

Here T is the transpose. The image of ${A_{ij}}^{-1}$ under γ_n was computed rather than the image of A_{ij} for the reason of getting a more convenient representation, in the sense that the image of ${A_{ij}}^{-1}$ involves only the indices i and j. It is then clear that the Gassner representation is reducible because there is an invariant subspace spanned by the vector $(1, \dots, 1)^T$, which is fixed by the action of this representation (T is the transpose). It is common knowledge that the Gassner representation is reducible as was shown in [2, p.121]. We will show that γ_n is the direct sum of a trivial representation and an irreducible representation called the reduced Gassner representation and denoted by $\overline{\gamma_n}$. Notice that for $j \neq n$, the last row of X_{ij} is $(0, \dots, 0, 1)$. Delete the last row and column to obtain an $n-1 \times n-1$ matrix Y_{ij}, where $Y_{ij} = I - \overline{P_{ij}}\ \overline{Q_{ij}}$. Here $\overline{P_{ij}}$ and $\overline{Q_{ij}}$ are the same as P_{ij} and Q_{ij} after deleting one zero from the last row of P_{ij}, and one zero from the last column of Q_{ij}, respectively. For $j = n$, put $Y_{in} = I - \overline{P_{in}}\ \overline{Q_{in}}$, where

$$\overline{P_{in}} = \big(\underbrace{(1-y_i)t \dots (1-y_i)t}_{i-1} \quad 1-y_it \quad \underbrace{1-y_i \dots 1-y_i}_{n-1-i}\big)^T$$

and

$$\overline{Q_{in}} = \big(\underbrace{0 \dots 0}_{i-1} \quad 1 \quad \underbrace{0 \dots 0}_{n-1-i}\big),$$

where $t = y_n$. By part (b) of Theorem 2, this is a choice for $\overline{P_{in}}$ and $\overline{Q_{in}}$ for which the matrix given by the inner product $(\overline{Q_{in}}\ \overline{P_{jn}})$ is equal to the matrix $(Q_{in}P_{jn})$. The details are found in [1, p.27] and [4, p.6]. Here Y_{ij} is the image of A_{ij}^{-1} under the reduced Gassner representation $\overline{\gamma_n}$. There is no explicit representation which is agreed to be “the” reduced Gassner representation of P_n.

Theorem 4. *Let G be a group and $\rho : G \to GL_n(\mathbb{C})$ a representation of degree n. Then ρ is a direct sum of a representation of degree $n-1$ and a trivial representation if and only if there are a row vector w and a column vector v fixed by $\rho(G)$ such that $wv = 1$.*

Proof. Here $GL_n(\mathbb{C})$ is considered as acting from the left on column vectors or acting from the right on row vectors.

$\Longrightarrow$ Without loss of generality, let

$$\rho(x) = \begin{pmatrix} \bar{\rho}(x) & 0 \\ 0 & 1 \end{pmatrix},$$

where $x \in G$. Then it is clear that the vectors e_n and f_n are fixed by $\rho(G)$, where $e_n = (0 \dots 1)$, $\quad f_n = (0 \dots 1)^T$ and $e_nf_n = 1$ (T is the transpose).

$\Longleftarrow$ Assume that there exist such row vectors and column vectors w and v fixed by $\rho(G)$ and $wv = 1$. Let M be an $n \times n$ invertible matrix such that $wM = e_n$, then

$$1 = wv = e_nM^{-1}v.$$

This implies that $M^{-1}v$ is a column vector of the form $(*, \dots, *, 1)^T$, where the $*$'s are some complex numbers. Now there exists an invertible matrix N such that $e_nN = e_n$ and $Nf_n = M^{-1}v$. To see this, we can choose N as the following matrix:

$$N = \begin{pmatrix} 1 & 0 & \dots & 0 & * \\ 0 & 1 & \dots & & * \\ \vdots & & \ddots & & \vdots \\ & & & 1 & * \\ 0 & \dots & & 0 & 1 \end{pmatrix},$$

where the last column of N is exactly the vector $M^{-1}v$. Then we get

$$(w)MN = e_n \quad \text{and} \quad MN(f_n) = v.$$

Consider the equivalent representation of ρ, namely $(MN)^{-1}\rho(MN)$; then for $x \in G$, we have

$$(MN)^{-1}\rho(x)(MN)(f_n) = f_n \quad \text{and} \quad (e_n)(MN)^{-1}\rho(x)(MN) = e_n.$$

It is then easy to observe that $(MN)^{-1}\rho(x)(MN)$ must be of the form

$$\begin{pmatrix} \bar{\rho}(x) & 0 \\ 0 & 1 \end{pmatrix},$$

where $\bar{\rho}$ is a representation of degree $n-1$. □

Lemma 5. *The Gassner representation γ_n is the direct sum of the reduced representation of degree $n-1$, denoted by $\overline{\gamma_n}$, and the trivial representation.*

Proof. In light of Theorem 4, it suffices to find a row vector and a column vector fixed by γ_n whose inner product is one. Let w be the row vector defined as follows:

$$w = \frac{1}{y_1 \dots y_n - 1}[y_2 \dots y_n(y_1 - 1), y_3 \dots y_n(y_2 - 1), \dots, y_n(y_{n-1} - 1), y_n - 1]$$

and

$$v = (1, \dots, 1)^T.$$

It is then clear, by direct calculation, that these vectors are fixed by γ_n. □

3. The restriction of the reduced Gassner representation to a certain normal free subgroup

To apply Theorem 2 for the reduced Gassner representation of P_n, we will construct for each $i = 1, \dots, n$ a free normal subgroup of rank $n-1$, namely, V_i. Let V_i be the subgroup generated by the elements

$$A_{1,i}, A_{2,i}, \dots, A_{i-1,i}, A_{i,i+1}, \dots, A_{in},$$

where $A_{i,j}$ are those generators of P_n. Recall that the image of A_{ij}^{-1} under the reduced Gassner representation is denoted by Y_{ij} , where $Y_{ij} = I - \overline{P_{ij}}\;\overline{Q_{ij}}$.

In other words, the generators of V_i are $A_{i,j}$ where $A_{i,j} = A_{j,i}$ whenever $i > j$ and $j = 1, 2, \dots, n$. It is known that V_i generates a free subgroup of P_n which is isomorphic to the subgroup V_n freely generated by $\{A_{1,n}, A_{2,n}, \dots, A_{n-1,n}\}$. This is intuitively clear because it is quite arbitrary how we assign indices to the braid "strings".

Let $\overline{\gamma_n} : V_i \to GL_{n-1}(\mathbb{Z}[y_1^{\pm 1}, \dots, y_n^{\pm 1}])$ and Y_{ij} be the image of the generator A_{ij}^{-1} under $\overline{\gamma_n}$. Then

$$Y_{ij} = I - \overline{P_{ij}}\;\overline{Q_{ij}}\ ,$$

where $\overline{Q_{ij}}\;\overline{P_{ik}} = Q_{ij}P_{ik}$.

Again all choices of $\overline{P_{ij}}$, $\overline{Q_{ij}}$, for which $(\overline{Q_{ij}}\;\overline{P_{ik}}) = (Q_{ij}P_{ik})$, give equivalent representations of V_i. Here $P_{ki} = P_{ik}$ and $\overline{P_{ki}} = \overline{P_{ik}}$ whenever $k > i$, and similarly for Q_{ij} and $\overline{Q_{ij}}$.

Direct calculations show that for $1 \leq i, j, k \leq n$, the matrix $(Q_{ij}P_{ik})$ given by the inner product is the following:

$$\left(\begin{array}{ccc|cccc}
1-y_1y_i & \dots & (1-y_{i-1})y_i & y_{i+1}-1 & y_{i+2}-1 & \dots & y_n-1 \\
1-y_1 & \dots & (1-y_{i-1})y_i & y_{i+1}-1 & y_{i+2}-1 & \dots & y_n-1 \\
1-y_1 & \dots & (1-y_{i-1})y_i & y_{i+1}-1 & y_{i+2}-1 & \dots & y_n-1 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
1-y_1 & \dots & 1-y_{i-1}y_i & y_{i+1}-1 & y_{i+2}-1 & \dots & y_n-1 \\
\hline
(y_1-1)y_i & \dots & (y_{i-1}-1)y_i & 1-y_iy_{i+1} & (1-y_{i+2})y_i & \dots & (1-y_n)y_i \\
(y_1-1)y_i & \dots & (y_{i-1}-1)y_i & 1-y_{i+1} & 1-y_iy_{i+2} & \dots & (1-y_n)y_i \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
(y_1-1)y_i & \dots & (y_{i-1}-1)y_i & 1-y_{i+1} & 1-y_{i+2} & \dots & 1-y_iy_n
\end{array}\right)$$

The upper left submatrix is an $(i-1) \times (i-1)$ matrix. More precisely, it is the matrix given by the inner product $(Q_{ji}P_{ki})$, where $1 \leq j, k \leq i-1$.

Lemma 6. *Let X_n be the $(n-1) \times (n-1)$ matrix $(Q_{ij}P_{ik})$. Then*

$$det(X_n) = (1-y_i)^{n-2}(1-y_1y_2\dots y_n).$$

Theorem 7. *Let i be an integer such that $1 \leq i \leq n$. Then $\overline{\gamma_n}$ is irreducible on V_i if and only if $(1-y_i)^{n-2}(1-y_1\dots y_n) \neq 0$.*

Proof. This directly follows by Lemma 6 and Theorem 2 (a). □

By Theorem 7, the reduced Gassner representation of P_n is irreducible. Recall that the indeterminates involved in defining this representation are independent and that irreducibility of a representation on a subgroup implies the irreducibility on the group itself.

Let $G_n(z)$ be the representation

$$P_n \to GL_{n-1}(\mathbb{C})$$

obtained by specializing $y_i \to z_i$ in the reduced Gassner representation $\overline{\gamma_n}$, where $z = (z_1, \dots, z_n)$; $z_1, \dots, z_n \in \mathbb{C}^*$.

Next in my work, I will consider $GL_{n-1}(\mathbb{C})$ acting from the right on row vectors. I will show that if $z = (z_1, \dots, z_n)$ and $z_1 \dots z_n = 1$, then there is a nonzero vector fixed by the specialization of the reduced Gassner representation $G_n(z)$.

Proposition 8. *If $z_1 \dots z_n = 1$, then the vector $[z_2 \dots z_{n-1}(z_1-1), z_3 \dots z_{n-1}(z_2-1), \dots, z_{n-1}-1]$ is fixed by $G_n(z)$.*

Proof. For simplicity, denote the above vector by v and for $i = 1, 2, \dots n-2$ define a_i and a_{n-1} as follows:

$$a_i = z_{i+1} \dots z_{n-1}(z_i - 1) \qquad \text{and} \qquad a_{n-1} = z_{n-1} - 1.$$

If $z_i = 1$ for every $i = 1, \dots, n-1$, then it is clear that, by our hypothesis, $z_n = 1$. In this case, $G_n(z)$ will be trivial. Therefore, we may assume that $z_i \neq 1$ for some $i = 1, \dots, n-1$.

Consider here Y_{ij} as the image of A_{ij}^{-1} under the complex specialization of the reduced Gassner representation of P_n. I will show that for every $1 \leq i < j \leq n$, we have

$$vY_{ij} = v.$$

Case 1. Consider Y_{ij} for $j \neq n$,

$$Y_{ij} = I - \overline{P_{ij}}\,\overline{Q_{ij}},$$

where

$$\overline{P_{ij}} =$$
$$\big(\underbrace{0\dots0}_{i-1} \quad z_j - 1 \quad (z_i-1)(1-z_j)\dots \quad (z_i-1)(1-z_j) \quad (z_i-1)(-z_j) \quad \underbrace{0\dots0}_{n-1-j}\big)^T$$

and

$$\overline{Q_{ij}} = \big(\underbrace{0\dots0}_{i-1} \quad -1 \quad 0\dots \quad 0 \quad 1 \quad \underbrace{0\dots0}_{n-1-j}\big).$$

$$vY_{ij} = v(I - \overline{P_{ij}}\,\overline{Q_{ij}}) = v - (v\overline{P_{ij}})\,\overline{Q_{ij}}. \tag{1}$$

Simple calculation shows that $v\overline{P_{ij}} = (1-z_j)(z_i-1)A$, where

$$\begin{aligned} A = &-z_{i+1}\dots z_{n-1} + z_{i+1}\dots z_{n-1} \\ &- z_{i+2}\dots z_{n-1} + z_{i+2}\dots z_{n-1} \\ &\vdots \\ &- z_j\dots z_{n-1} + z_j\dots z_{n-1}. \end{aligned}$$

Simplifying, we obtain that

$$A = 0.$$

By (1), we get

$$vY_{ij} = v.$$

Case 2. Let $j = n$ and consider $Y_{in} = I - \overline{P_{in}}\,\overline{Q_{in}}$, where

$$\overline{P_{in}} = \big(\underbrace{(1-z_i)z_n\dots(1-z_i)z_n}_{i-1} \quad 1 - z_iz_n \quad \underbrace{1-z_i\dots1-z_i}_{n-1-i}\big)^T$$

and

$$\overline{Q_{in}} = \big(\underbrace{0\dots0}_{i-1} \quad 1 \quad \underbrace{0\dots0}_{n-1-i}\big).$$

Again

$$vY_{in} = v - (v\overline{P_{in}})\,\overline{Q_{in}}. \tag{2}$$

By similar calculations as in case 1, we get $v\overline{P_{in}} = (1 - z_i)B$, where

$$\begin{aligned} B = {} & z_1 \dots z_{n-1} z_n - z_2 \dots z_{n-1} z_n \\ & + z_2 \dots z_{n-1} z_n - z_3 \dots z_{n-1} z_n \\ & \vdots \\ & - z_{i+1} \dots z_{n-1} + z_i \dots z_n \\ & + z_{i+1} \dots z_{n-1} - z_{i+2} \dots z_{n-1} \\ & \vdots \\ & + z_{n-1} - 1. \end{aligned}$$

Since, by our hypothesis, $z_1 \dots z_n = 1$, it follows that, after simplifying,

$$B = 0.$$

Hence by (2), we get $vY_{in} = v$.

Therefore, by treating all the possible cases, we have proved that for $1 \leq i < j \leq n$

$$vY_{ij} = v. \qquad \square$$

Recall that irreducibility of a representation of a group in $GL_{n-1}(\mathbb{C})$ means that there are no invariant subspaces under the action of this representation. Now we present the proof of the main theorem.

4. Proof of Main Theorem

Suppose, to get a contradiction, that $z_1 \dots z_n = 1$. Then, by Proposition 8, there is a nonzero vector v which is fixed by $G_n(z)$. This contradicts the irreducibility of $G_n(z)$.

Now assume that $z_1 \dots z_n \neq 1$. I will show that $G_n(z)$ is irreducible. Since $z_1 \dots z_n \neq 1$, it follows that at least one $z_i \neq 1$ for some $i = 1, \dots, n$. Then we have

$$(1 - z_i)^{n-2}(1 - z_1 \dots z_n) \neq 0.$$

Considering the corresponding subgroup V_i, we find out that $G_n(z)$ is irreducible on V_i by Theorem 7. Hence irreducibility on P_n follows immediately. $\square$

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