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GROUP HOMOMORPHISMS INDUCING mod-p COHOMOLOGY MONOMORPHISMS

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Abstract. Let $f\colon G \to K$ be a homomorphism of p-groups such that $f^{(n)}\colon H^n(K, \mathbf{Z}/p) \to H^n(G, \mathbf{Z}/p)$ is injective, for $1 \le n \le 2$. We prove that the non-bijectivity of f implies the existence of a quotient L of G containing K as a proper direct factor. This gives a refined proof of a result of Evens, which asserts that f is bijective if $f^{(1)}$ is.

Let p be a prime number and let $\mathbf{Z}/p$ be the prime field of p elements. For every p-group K, let us denote by $H^*(K)$ the mod-p cohomology of K.

Let $f\colon G \to K$ be a homomorphism of p-groups such that $f^{(n)}\colon H^n(K) \to H^n(G)$ is injective, for $1 \le n \le 2$. We shall give a refined proof of the following result of Evens.

Theorem A (Evens [1, Th. 7.2.4])**.** *If $f^{(1)}$ is bijective, then so is f.*

Further, we prove

Theorem B. *If f is not bijective, then there exists a quotient L of G containing K as a proper direct factor (i.e., $L = J \times K$, with $J \ne \{1\}$).*

Note that the maps $H^1(K) \xrightarrow{\mathrm{Inf}_1(L,K)} H^1(L)$ and $H^1(L) \xrightarrow{\mathrm{Inf}_1(G,L)} H^1(G)$ are injective, and $\mathrm{Im}\,\mathrm{Inf}_1(L,K)$ is a proper subgroup of $H^1(L)$ (since $J \ne \{1\}$), so, by Theorem B, the non-bijectivity of f implies that

$$\dim_{\mathbf{Z}/p} H^1(K) < \dim_{\mathbf{Z}/p} H^1(L) \le \dim_{\mathbf{Z}/p} H^1(G),$$

hence $f^{(1)}$ is not bijective. We obtain then an alternative proof of Theorem A.

Proof of Theorem B. We first prove that f is surjective. Set $K' = \mathrm{Im}\, f$. So f factors through $G \xrightarrow{f} K' \hookrightarrow K$. Since $f^{(1)} = (H^1(K) \xrightarrow{\mathrm{Res}} H^1(K') \xrightarrow{f^*} H^1(G))$ is injective, it follows that $\mathrm{Res}\colon H^1(K) \to H^1(K')$ is injective. This implies $K = K'$, so f is surjective.

Consider the extension

$$1 \to \mathrm{Ker}\, f \to G \xrightarrow{f} K \to 1.$$

Let p^m be the order of $\mathrm{Ker}\, f$ ($m \ge 1$, since f is not bijective). The above extension

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can then be obtained by successive central extensions

$$(G_i) \qquad 0 \to \mathbf{Z}/p \to G_i \xrightarrow{p_i} G_{i-1} \to 1,$$

$1 \leq i \leq m$, with $G_0 = K$, $G_m = G$, $p_1 \circ \cdots \circ p_m = f$.

Let $z \in H^2(K)$ be the factor set of the extension (G_1), viewed as a cohomology class of $H^2(K)$; then $z \in \operatorname{Ker} \operatorname{Inf}_2(G_1, K)$. Since $f^{(2)} = \operatorname{Inf}_2(G, K) = \operatorname{Inf}_2(G, G_1) \circ \operatorname{Inf}_2(G_1, K)$, it follows from the injectivity of $f^{(2)}$ that $\operatorname{Inf}_2(G_1, K)$ is also injective. Hence $z = 0$, so $G_1 = \mathbf{Z}/p \times K$. The theorem follows.

References

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