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$\mathcal{L}$-CLASSES ON PSEUDOMANIFOLDS WITH ONE SINGULAR STRATUM

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ABSTRACT. We study the index theorem and Chern character of an admissible pseudomanifold $X^\dagger$ with one singular stratum. Under a condition on the link, we give a de Rham type realization of the Goresky-MacPherson-Siegel $\mathcal{L}$-classes on $X^\dagger$ in terms of curvature forms and eta invariant of the link.

0. INTRODUCTION

In [CST] and [MW1] Connes-Sullivan-Teleman and Moscovici-Wu solved a long-standing problem of recovering the topological Pontryagin classes from local data. In [MW1] and [MW2], using the finite propagation speed property in the unbounded picture, Moscovici-Wu gave a realization of the $\mathcal{L}$-classes, hence of the Pontryagin classes, of the topological manifold in terms of Alexander-Spanier cycles by looking at the "straight" Chern character of [CM].

In this paper, we will generalize the result in [MW1] to a particular type of pseudomanifolds, namely $X^\dagger = M \cup (c^\dagger(L) \times N)$. As in [C1] and [GM], for spaces with singularities, we work with "characteristic classes" in homology rather than cohomology. In [GM] Goresky and MacPherson defined $\mathcal{L}$-classes for pseudomanifolds with even codimensional strata. By the work of Siegel [Si], one can extend this definition to Witt spaces. Our approach differs from [MW1] in the sense that we relate the "straight" Chern character of D to $\mathcal{L}$-class directly. This asserts that (Theorem 3.4) the Goresky-MacPherson-Siegel $\mathcal{L}$-class of $X^\dagger$ (with $2L$ having zero oriented cobordism) is represented by the following cycle:

$$L_*(X^\dagger; \rho_{X^\dagger}) = 2^{2m'} \mathcal{L}_{m'}(R(g^M)) \oplus -\delta_\ell \, \eta(L) \, 2^{2n'} \mathcal{L}_{n'}(R(g^N)),$$

where $m' = \frac{m-*}{4}$, $n' = \frac{n-*}{4}$ and $\delta_\ell = \frac{1}{2}(1-(-1)^\ell)$. Furthermore, we will show that the Chern character of D coincides with the "straight" Chern character of D.

In [BC] and [C2], Bismut and Cheeger defined $\mathcal{L}$-classes by means of the index pairing. Theorem 4.1 below shows that the Bismut-Cheeger $\mathcal{L}$-classes and Goresky-MacPherson-Siegel $\mathcal{L}$-classes are the same (up to constants) for spaces $X^\dagger$ defined above.

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1. Preliminaries

1.1. **Pseudomanifolds.** For your reference, let's recall some definitions in [C1]. An m-dimensional pseudomanifold Y^m is a finite simplicial complex such that every point is contained in a closed m-simplex and every $(m-1)$-simplex is a face of either one or two m-simplices. We also endow Y^m with a metric, which determines a distance function ρ_Y, and assume that $Y \setminus \sum^{m-2}$ is a flat manifold (in the induced metric) where $\sum^{m-2}$ is the $(m-2)$-skeleton associated with a triangulation. A pseudomanifold is called admissible [C1, p.127] if the middle L^2-cohomology group of even dimensional links, in the Riemannian handle decomposition, vanishes.

After the construction of Fredholm module in Section 2, we will consider a particular type of pseudomanifolds:

Let M be a smooth, oriented, compact and connected m-dimensional manifold with boundary $\partial M = L \times N$ where L and N are smooth, oriented and compact manifolds (without boundary) of dimensions ℓ and n respectively, and either ℓ is odd or $\mathrm{H}^{\frac{\ell}{2}}(L) = 0$.

In this paper, we will use the following notation:

$$
\begin{array}{rcll}
c_{0,\infty}(L) & = & (0,\infty) \times L & \text{infinite cone with link } L, \\
c(L) & = & (0,1) \times L & \text{cone with link } L, \\
\dagger & = & \text{the tip of the cone,} & \\
c^\dagger(L) & = & c(L) \cup \{\dagger\} & \text{completed cone with link } L, \\
X & = & M \cup\ (c(L) \times N) & \text{regular part,} \\
X^\dagger & = & M \cup\ (c^\dagger(L) \times N) & \text{pseudomanifold with one singular stratum.}
\end{array}
$$

We will endow $X^\dagger$ with a metric g (not necessary flat on X) such that:

(i) g is a measurable metric on $X^\dagger$;
(ii) $g|_M$ is a smooth metric on M and is a product near ∂M;
(iii) $g|_{c^\dagger(L)\times N} = (dr^2 + \psi(r)^2 g^L) \oplus g^N$, where g^L and g^N are smooth metrics on L and N respectively, and $\psi : [0,1] \to [0,1]$ is a C^∞ function such that

$$\psi(r) = \begin{cases} r, & r \in [0, \frac{2}{3}], \\ 1, & r \in [\frac{3}{4}, 1], \end{cases}$$

and $\psi(r) \neq 0$ for $r > 0$.

Clearly, this is a Lipschitz metric.

In order not to change the metric (i.e. $\rho_{X^\dagger}|_{X\times X} = \rho_X$), L must be connected. Otherwise, we will need to add a cone on each component of L. In other words, we have to consider the normalization of the space [GM, p.151]. So in the rest of this paper, we will assume that L and N are connected. In this case, $X^\dagger$ is the metric completion of X.

1.2. **Sullivan complex.** In the presence of singularities it is more convenient to use the Sullivan complex ([BC], [MW2] and [Su]). Let's recall the corresponding results [MW2] for the space $X^\dagger$. Let $pr : c(L) \times N \to N$ be the projection onto the second factor and $j : L \times N \to M$ be the inclusion map. The complex of stratified differential forms on $X^\dagger$ is:

$$\Omega^*(X^\dagger)_{SA} = \{\omega \in \Omega^*(X) : \omega|_{c(L)\times N} = pr^*(\tilde{\omega}), \quad \tilde{\omega} \in \Omega^*(N)\}.$$

$\Omega^0(X^\dagger)_{SA}$ is an algebra, which will also be denoted by $C^\infty_{SA}(X^\dagger)$. The differentials are the usual ones. From [MW2], $H^*(X^\dagger) \cong H_*(\Omega^*(X^\dagger)_{SA}, d)$. For homology, there is also a chain complex of de Rham type, namely

$$\Omega_q(X^\dagger)_{SA} = \Omega^{m-q}(M) \oplus \Omega^{n-q}(N)$$

with the boundary operator $\partial : \Omega_q(X^\dagger)_{SA} \to \Omega_{q-1}(X^\dagger)_{SA}$ given by

$$\partial(\omega_1, \omega_2) = \left((-1)^q d\omega_1, (-1)^q d\omega_2 + \int_L j^*\omega_1\right).$$

Again from [MW2], $H_*(X^\dagger) \cong H_*(\Omega_*(X^\dagger)_{SA}, \partial)$.

For these complexes, there is a pairing

$$\begin{array}{rcl} \Omega_q(X^\dagger)_{SA} \otimes \Omega^q(X^\dagger)_{SA} & \longrightarrow & \mathbb{C}, \\ (a,b) \otimes c & \longmapsto & \displaystyle\int_M a \wedge c + \int_N b \wedge c \ , \end{array}$$

which induces a pairing on the corresponding homology and cohomology.

1.3. **Goresky-MacPherson-Siegel $\mathcal{L}$-class.** Recall that [Si, p.1068] a pseudomanifold is a Witt-space if every even dimensional link $\mathbf{L}^\ell$ of an odd codimensional intrinsic stratum satisfies $IH^{\bar m}_{\frac{\ell}{2}}(\mathbf{L};\mathbb{Q}) = 0$. Let V be a Witt-space which is also a compact subset of some C^∞-manifold V'. If $\dim V \neq 4k$, then we define the signature σ of V to be 0. If $\dim V = 4k$, then let $\bar m = (0,0,1,1,2,2,\ldots,2k-1)$. By [Si, Theorem 3.4], there is a non-degenerate rational pairing

$$IH^{\bar m}_i(V;\mathbb{Q}) \times IH^{\bar m}_j(V;\mathbb{Q}) \to \mathbb{Q}$$

for $i + j = \dim V$, $i, j \geq 0$. In this case, the signature σ of V is defined as the signature of the associated quadratic form.

As in [GM, p.158], a continuous map $f : V \to S^k$ is called transverse if

- **(a):** f is the restriction of a C^∞-map $\tilde f : U \to S^k$ for some neighborhood U of V in V',
- **(b):** $\tilde f$ is transverse regular to the north pole $p \in S^k$,
- **(c):** $\tilde f^{-1}(p)$ is transverse to each stratum of V.

As there is such a representative in each homotopy class and the signature is cobordism invariant [GM, p.158], [Si, Theorem 2.1], one can define

$$\begin{array}{rcl} \theta : [V, S^k] & \longrightarrow & \mathbb{Z}, \\ [f] & \longmapsto & \sigma(f^{-1}(p)). \end{array}$$

The Goresky-MacPherson-Siegel $\mathcal{L}$-class, $\mathcal{L}_k(V) \in H_k(V;\mathbb{R})$, is defined as the homomorphism

$$\theta \otimes I : H^k(V;\mathbb{R}) \to \mathbb{R}$$

where we have identified $[V, S^k] \otimes \mathbb{R} \cong H^k(V;\mathbb{R})$ when $2k > m+1$.

We can remove the assumption $2k > m+1$ by crossing V with a sphere as in [GM, p.158] and [MS].

2. Signature operator and K-cycle

2.1. **Finite propagation speed.** Let Y be an admissible Riemannian pseudo-manifold. In this subsection, the domains of the operators are:

$$\begin{aligned}\mathrm{Dom}(d) = \{\alpha \in \Gamma^\infty : \alpha, d\alpha \in L^2\},\\ \mathrm{Dom}(\delta) = \{\alpha \in \Gamma^\infty : \alpha, \delta\alpha \in L^2\}.\end{aligned}$$

In the next subsection, we will show that it does not matter which domain we use.

By [C1], $\overline{d}^* = \overline{\delta}$. Then by [H1, Lemma 4.3], $D := \overline{d} + \overline{\delta}$ is self-adjoint with domain $\mathrm{Dom}(\overline{d}) \cap \mathrm{Dom}(\overline{\delta})$.

Let $f \in C(Y)$ act on $L^2(\wedge^* T(Y \setminus \sum))$ by multiplication.

Lemma 2.1. *Let $h \in C^{Lip}(Y)$; then $h \cdot Dom(D) \subset Dom(D)$.*

Proof. Let $\omega \in \mathrm{Dom}(D)$ and J_ϵ be the mollifier corresponding to a bump function Φ. So $J_\epsilon(h\omega)$ is smooth and $J_\epsilon(h\omega) \stackrel{L^2}{\to} h\omega$. Since

$$\begin{aligned} d(h\omega) &= dh \wedge \omega + hd\omega, \\ \delta(h\omega) &= *(dh \wedge *\omega) + h\delta\omega, \end{aligned}$$

$\|f\|_\infty, \|dh\|_\infty \leq 1$, $\omega, d\omega, \delta\omega \in L^2$ and by using partition of unity, we have

$$\begin{aligned} J_\epsilon(h\omega)(x) &= \frac{1}{\epsilon^n}\int \Phi(\frac{x-y}{\epsilon})h(y)\omega(y)dy \\ &= \frac{1}{\epsilon^n}\int \Phi(\frac{y}{\epsilon})h(x-y)\omega(x-y)dy \end{aligned}$$

in local coordinate patch.

Then $J_\epsilon(h\omega), d(J_\epsilon(h\omega)), \delta(J_\epsilon(h\omega)), d(h\omega), \delta(h\omega) \in L^2$.

By Friedrichs Lemma [T, p.114],

$$dJ_\epsilon(h\omega) \stackrel{L^2}{\to} d(h\omega), \qquad \delta J_\epsilon(h\omega) \stackrel{L^2}{\to} \delta(h\omega).$$

Hence $h\omega \in \mathrm{Dom}(D)$. ☐

Proposition 2.2. *D has finite propagation speed with respect to $C(Y)$, i.e.*

$$\forall t \in \mathbb{R}, \qquad supp(e^{itD}) \subset \{(x,y) \in Y \times Y : \rho_Y(x,y) \leq |t|\}.$$

Also, let $f \in \mathcal{S}(\mathbb{R})$ such that $supp\hat{f} \subset [-\alpha, \alpha]$ for some $\alpha > 0$; then

$$supp(f(D)) \subset \{(x,y) \in Y \times Y : \rho_Y(x,y) \leq \alpha\}.$$

Proof. As ρ_Y is a Lipschitz function on $Y \setminus \sum^{m-2}$, the results follow from [H2, Lemma 1.10] as in [H2, Corollary 1.11]. ☐

In the remainder of this paper, we will assume m to be even, unless otherwise stated.

Remark 2.3. So far, the results for Y in this section are still true when Y endows a metric which is quasi-isometric to a flat metric.

2.2. **Essential self-adjointness of signature operator.** Due to some technicalities in the rest of this paper, we will investigate the essential self-adjointness of the twisted signature operator on $X^\dagger$. Assume dim$X$ is even; then $\partial X = L \times N$ is odd dimensional. Let (E, ∇) be a Hermitian vector bundle on X with a unitary connection such that its restriction to $c(L) \times N$ is pulled back from N.

We will consider the scaling of the metric in the conical direction as follows:

$$g|_{c(L)\times N} = \left(\frac{dr^2}{\epsilon} + \psi(r)^2 g^L\right) \oplus g^N$$

for $\epsilon > 0$.

In the following we will use the standard domain, specifically $\mathrm{Dom}(D) = \{\alpha \in \Gamma_c^\infty(X)\}$.

To study the self-adjointness of a operator D, we will examine the deficiency indices $n_\pm(\mathrm{D}) = \dim \mathrm{Ker}(\mathrm{D} \mp iI)$. Let us recall a proposition from [L1].

Proposition 2.4 ([L1], Corollary 2.2)**.** *Let $\mathcal{M}_i$ be an oriented Riemannian manifold and D_i be a generalized Dirac operator over $\mathcal{M}_i$, $i = 1, 2$. Let U_i be an open subset of $\mathcal{M}_i$, $i = 1, 2$. Suppose $\mathcal{M}_1 \setminus U_1$ and $\mathcal{M}_2 \setminus U_2$ are complete manifolds with compact boundary and there exists an isometry from $\gamma : U_1 \to U_2$ which lifts to an isomorphism of Clifford structure. Then $n_\pm(D_1) = n_\pm(D_2)$.*

For easy reference, we will add subscripts to operators to indicate the underlying manifold and vector bundle.

Proposition 2.5. *There exists $\delta > 0$ (independent of E) such that $\forall \epsilon \in (0, \delta)$, the twisted signature operator $D_{X,E}$ is essentially self-adjoint.*

Proof. We will divide this into two cases.

Case 1: ℓ is odd and n is even.

On $c(L) \times N$, $D_{X,E} \simeq D_{c(L)} \hat{\otimes} I + I \hat{\otimes} D_{N,E}$.

For sufficiently small ϵ, by [BS, Lemma 5.4], $D_{c_{0,\infty}(L)}$ is essentially self-adjoint. Hence, by [RS, Theorem VIII.33], $D_{c_{0,\infty}(L)\times N,E}$ is essentially self-adjoint. Therefore,

$$n_\pm(D_{c_{0,\infty}(L)\times N,E}) = 0.$$

Then by Proposition 2.4,

$$n_\pm(D_{X,E}) = n_\pm(D_{c_{0,\infty}(L)\times N,E}) = 0.$$

Therefore, $D_{X,E}$ is essentially self-adjoint for sufficiently small ϵ.

Case 2: ℓ is even and n is odd.

Since $\dim(c_{0,\infty}(L))$ is odd, the signature operator splits as

$$D_{c_{0,\infty}(L)} = D^+_{c_{0,\infty}(L)} \oplus D^-_{c_{0,\infty}(L)}.$$

Let λ_{ω_L} be the Clifford multiplication by $i^{[\frac{l+1}{2}]} e_1 \cdots e_l$ where $(e_1, \ \ldots, e_l)$ is an oriented orthonormal frame for TL.

By [L1, Lemma 1.2, Proposition 4.1, Proposition 5.3],

$$\begin{aligned} n_\pm(D^+_{c_{0,\infty}(L)}) &= n_\mp(D^-_{c_{0,\infty}(L)}) \\ &= \dim(\mathrm{Ker}(\lambda_{\omega_L} \mp I) \cap \ \mathrm{Ker}P) + \sum_{0<\lambda<\frac{1}{2}} \dim \mathrm{Ker}(P - \lambda I) \end{aligned}$$

for some operator P.

By [BL, Lemma 2.2], Ker$P = H^{\frac{\ell}{2}}(L) = 0$.
By [BL, Corollary 2.3],

$$\sum_{0<\lambda<\frac{1}{2}} \dim \operatorname{Ker}(P - \lambda I) = 0 \qquad \text{for sufficiently small } \epsilon.$$

Thus, $D^{\pm}_{c_{0,\infty}(L)}$ is essentially self-adjoint and so is $D_{c_{0,\infty}(L)}$. Notice that

$$D_{c_{0,\infty}(L)\times N,E} = D_{c_{0,\infty}(L)} \otimes I + \phi \otimes D_{N,E}, \qquad \text{where } \phi = \begin{cases} 1 & \text{on } \Omega^{even}, \\ -1 & \text{on } \Omega^{odd}. \end{cases}$$

Then as in the proof of [RS, Theorem VIII.33], $D_{c_{0,\infty}(L)\times N,E}$ is essentially self-adjoint. Thus,

$$n_{\pm}(D_{c_{0,\infty}(L)\times N,E}) = 0.$$

By Proposition 2.4,

$$n_{\pm}(D_{X,E}) = n_{\pm}(D_{c_{0,\infty}(L)\times N,E}) = 0.$$

Hence the result follows. □

2.3. **Singular elliptic estimate.** In the remaining of this paper, we will assume a scaling on the conical metric such that there are unique self-adjoint extensions for the signature operator on X and the twisted signature operator on X as well as the twisted signature operator on $c_{0,\infty}(L) \times N$. These facts will be used to remove the effect of small eigenvalues. By abuse of notation, we will use the same symbol to denote the self-adjoint extensions of the operators. As in Section 2.2, dim$X$ is even.

Let (E, ∇) be a Hermitian vector bundle on X with unitary connection such that its restriction to $c(L) \times N$ is pulled back from N.

To furnish the computation on heat kernels, we need to recall a definition from [L2]. Let $\mathcal{M}$ be a Riemannian manifold and $U \subset \mathcal{M}$ an open subset with compact boundary. Let P_0 be a symmetric differential operator of order μ defined on a bundle over $\mathcal{M}$. Assume there exists a closed self-adjoint extension P of P_0 such that Dom(P) is invariant under multiplication by functions $\varphi \in C^{\infty}_{U}(M)$ satisfying $\varphi|_U \equiv 1$. (This is always true if $\mathcal{M} \setminus U$ is compact.)

Let $\mathcal{K}D(P, U) := \{s \in \operatorname{Dom}(P) \ : \ \operatorname{supp} s \subset U, \ \operatorname{dist}(\operatorname{supp} s, \partial U) > 0\}$.

Definition 2.6 ([L2], p.41)**.** P satisfies the singular elliptic estimate (SE) on U if $\exists \varrho \in L^2_{loc}(\mathcal{M}) \cap C(\mathcal{M}), \varrho > 0, \varrho|_U \in L^2(U)$ and $\ell \in \mathbb{R}^+$ such that for $x \in U$ and $s \in \mathcal{K}D(P^{\ell}, U)$,

$$|s(x)| \le \varrho(x) \left(||s||_{L^2(U,E)} + ||P^{\ell} s||_{L^2(U,E)} \right).$$

The importance of this concept lies in the following theorem.

Theorem 2.7 ([L2])**.** *Let $\mathcal{M}_i, U_i, P_{i,0}$ and P_i, $i = 1, 2$ as above. Assume there is an isometry $F : U_1 \to U_2$, which lifts to a bundle isometry $F_* : E_1|_{U_1} \to E_1|_{U_2}$ such that $P_{1,0} = F_*^{-1} \circ P_{2,0} \circ F_*$. We will identify U_1 with U_2 and denote it by U. We choose an open subset $W \subset U$, with smooth compact boundary such that $\overline{W} \subset U$ and $U \setminus W$ is relatively compact. If P_1 and P_2 satisfy (SE) over W, then $\forall N > 0, \exists C > 0$ such that for $x, y \in W$,*

$$\left| \left((P_1^2)^k e^{-tP_1^2} \right)(x, y) - \left((P_2^2)^k e^{-tP_2^2} \right)(x, y) \right| \le C \varrho(x) \varrho(y) t^N$$

where $k \in \mathbb{Z}_+ \cup \{0\}$.

In order to establish singular elliptic estimate for X, we need to recall some notation from [L2].

Let $\tilde{\rho} : X \to \mathbb{R}$ be a smooth function such that

$$\tilde{\rho}|_M = 1,$$
$$\tilde{\rho}((r,x,y)) = r \qquad \text{on } c_{0,\frac{1}{2}}(L) \times N.$$

Let $\mathcal{H}$ be a Hilbert space, $T > 0$ be a self-adjoint operator on $\mathcal{H}$ and

$$\mathcal{D}^\infty(T) := \bigcap_{k \geq 1} \mathrm{Dom}(T^k).$$

For $x, y \in \mathcal{D}^\infty(T)$ and $s \in \mathbb{R}$,

$$(x,y)_s := (T^s x, T^s y).$$

Let $\mathcal{H}^s_T$ be the completion of $\mathcal{D}^\infty(T)$ with respect to $\|\cdot\|_s$. Then by [L2, Lemma 1.2.1], $\Delta_L = (r\frac{\partial}{\partial r})^t (r\frac{\partial}{\partial r}) + D_L^2 \geq \frac{1}{4}$ and $\Delta = \Delta_L \otimes I + I \otimes D^2_{N,E}$ is essentially self-adjoint on $C^\infty_c(c_{0,\infty}(L)) \otimes C^\infty(N,E)$.

Define

$$\mathcal{H}^{s,0}(c_{0,\infty(L)\times N}, E) := \mathcal{H}^{\frac{s}{2}}_{\Delta},$$
$$\mathcal{H}^{s,\gamma}(c_{0,\infty(L)\times N}, E) := r^\gamma \mathcal{H}^{s,0}(c_{0,\infty(L)\times N}, E)$$
with scalar product $(f,g)_{s,\gamma} := (r^{-\gamma} f, r^{-\gamma} g)_s$.

Now let $\Delta_{X,E}$ be a non-negative elliptic differential operator of order 2 on X such that

$$\Delta_{X,E}|_M \geq c_1 \qquad \text{for some } c_1 > 0$$
and
$$\Delta_{X,E}|_{c(L)\times N} = \Delta.$$

By [L2, Corollary 2.2], $\Delta_{X,E}$ is essentially self-adjoint on $C^\infty_c(X,E)$.

Define

$$\mathcal{K}^{s,0}(X,E) := \mathcal{H}^{\frac{s}{2}}_{\Delta_{X,E}},$$
$$\mathcal{K}^{s,\gamma}(X,E) := \tilde{\rho}^\gamma \mathcal{K}^{s,0}(X,E)$$
with scalar product $(f,g)_{s,\gamma} := (\tilde{\rho}^{-\gamma} f, \tilde{\rho}^{-\gamma} g)_s$.

In [L2] Lesch proved the following lemma in the case when N is equal to a point. We shall prove this in a more general context.

Lemma 2.8. *There exist constants C and $\mu > 0$ such that*

$$Dom(D^m_{X,E}) \subset \mathcal{K}^{m,\mu}(X,E)$$

and $\forall f \in Dom(D^m_{X,E})$,

$$\|f\|_{m,\mu} \leq C\,(\|f\|_{0,0} + \|D^m_{X,E} f\|_{0,0}).$$

Proof. Let $\varphi \in C^\infty_o(\mathbb{R})$ such that $\varphi \equiv 1$ near 0.

By [L2, Proposition 1.3.19], the case when N is equal to a point, there exists $\mu > 0$ such that $\varphi\, \mathrm{Dom}(D^m_{c(L)}) \subset \mathcal{H}^{m,\mu}(c_{0,\infty}(L))$. So,

$$\varphi\, \mathrm{Dom}(D^m_{c(L)}) \otimes \mathrm{Dom}(D^m_{N,E}) \subset \mathcal{H}^{m,\mu}(c_{0,\infty}(L) \otimes N, E).$$

Hence, $\mathrm{Dom}(D^m_{X,E}) \subset \mathcal{K}^{m,\mu}(X,E)$.

The inequality in the lemma is now equivalent to the assertion that $\mathrm{Dom}(D^m_{X,E}) \hookrightarrow \mathcal{K}^{m,\mu}(X,E)$ is continuous. This follows from Closed Graph Theorem and the fact that the Sobolev norms in $\mathrm{Dom}(D^m_{X,E})$ and $\mathcal{K}^{m,\mu}(X,E)$ are stronger than the L^2-norm. □

Proposition 2.9. *There exist constants* C *and* $\mu > 0$ *such that* $\forall f \in Dom(D^m_{X,E})$,

$$|f(x)| \le C\tilde{\rho}(x)^{\mu - \frac{1}{2}}(\|f\| + \|D^m_{X,E} f\|).$$

That is, $D_{X,E}$ *satisfies the singular elliptic estimate (Definition 2.6).*

Proof. Follows from Lemma 2.8 and the corresponding estimate in model cone [L2, Cor. 1.2.9]. □

2.4. **The construction of K-cycle.** Let $\mathcal{H} = L^2(\wedge^* T(Y \setminus \sum)) = \mathcal{H}^+ \oplus \mathcal{H}^-$ with grading induced by $\epsilon = i^{p(p-1)+\frac{m}{2}} *$ and $*\mathrm{Dom}(D) \subset \mathrm{Dom}(D)$, $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$.

$A = C(Y) \overset{dense}{\supset} \mathfrak{A} = C^{Lip}(Y)$. Clearly $\mathfrak{A}$ is closed under holomorphic calculus of A and, by Lemma 2.1, leaves $\mathrm{Dom}(D)$ invariant and $\|[D, a]\| < \infty, \forall a \in \mathfrak{A}$.

Proposition 2.10. $(\mathcal{H}, D)$ *is an unbounded* p*-summable Fredholm module for* $p > m$. *Similar result holds for* $X^\dagger$.

Proof. By [G], $\overline{\Delta} = \overline{d}\,\overline{d}^* + \overline{\delta}\,\overline{\delta}^* = D^2$ is self-adjoint. By [C2, Lemma 7.1], D^2 has a discrete spectrum with finite multiplicities. By [C2, Theorem 7.2], we have the following asymptotic expansion of the heat kernel:

$$\mathrm{Tr}\,(e^{-tD^2}) = \int_X \mathrm{tr}\; e^{-tD^2}(x,x) \quad \sim \quad c_m + \sum_{j=0}^{m-2} c_j t^{-\frac{m}{2}+\frac{j}{2}} + \mathrm{O}(t^{-\frac{1}{2}}).$$

i.e.

$$\int_X \mathrm{tr}\; e^{-tD^2}(x,x) \quad \sim \quad t^{-\frac{m}{2}} c_0 + \mathrm{O}(t^{-\frac{m-1}{2}}).$$

By Karamata Theorem [BGV, p.94],

$$\begin{aligned} N(\lambda) &\sim \frac{c_0}{\Gamma(\frac{m+2}{2})}\lambda^{\frac{m}{2}} \\ \lambda_j &\sim C j^{\frac{2}{m}} \\ \mathrm{Tr}\,((1+D^2)^{-\frac{p}{2}}) &< \infty \qquad \text{for } p > m. \end{aligned}$$

By Proposition 2.9 and Theorem 2.7, one can obtain the asymptotic expansion on $X^\dagger$. Then result for $X^\dagger$ follows as above. □

3. "Straight" Chern character and $\mathcal{L}$-class

3.1. **"Straight" Chern character.** Since we have a Fredholm module for the signature operator, we can repeat the construction of the "straight" Chern character as in [CM] and [MW1]. Let's recall their construction:

Let $\overline{u}$ be an even smooth function on $\mathbb{R}$ such that $\overline{v}(x) = 1 - x^2\overline{u}(x)$ is Schwartz and both $\overline{u}$ and $\overline{v}$ have Fourier transforms supported on $(-\frac{1}{4}, \frac{1}{4})$. As $\overline{u}$ and $\overline{v}$ are even,

$$\overline{u}(x) = u(x^2), \qquad \overline{v}(x) = v(x^2)$$

for some smooth functions u and v.

Clearly v is also Schwartz and so is

$$w(x) = v(x)(1 + v(x))u(x).$$

Let $\gamma = i^{[\frac{m+1}{2}]}(-1)^{p(m-p)+\frac{p(p+1)}{2}} *_p : L_2(\Omega^p(X)) \to L_2(\Omega^{m-p}(X))$.

Then we consider the idempotent

$$P(tD) = \begin{pmatrix} (v(t^2D^2))^2\gamma & w(t^2D^2)\cdot tD\gamma \\ -v(t^2D^2)\cdot tD\gamma & (v(t^2D^2))^2\gamma \end{pmatrix} + \begin{pmatrix} \frac{1-\gamma}{2} & 0 \\ 0 & \frac{1-\gamma}{2} \end{pmatrix}$$

and define an Alexander-Spanier cycle $\Lambda_*(tD)$ as follows:

Let $f^0, \dots, f^{2q} \in C(X^\dagger)$,

$$q = 0,\ \ \Lambda_0(tD)(f^0) := \frac{1}{2}\mathrm{Tr}\left(P(tD)f^0 - \begin{pmatrix} \frac{1-\gamma}{2} & 0 \\ 0 & \frac{1-\gamma}{2} \end{pmatrix} f^0\right),$$

$$q > 0,\ \ \Lambda_{2q}(tD)(f^0\otimes\cdots\otimes f^{2q})$$

$$:= \frac{(2\pi i)^q}{q!(2q+1)2}\mathrm{Tr}\left(\sum_{\sigma\in S_{2q+1}} \mathrm{sgn}(\sigma)P(tD)f^{\sigma(0)}\cdots P(tD)f^{\sigma(2q)}\right),$$

and $\overline{ch}_{2q}(D) := \lim_{t\to 0}\Lambda_{2q}(tD)$.

By using the finite propagation speed, as in [MW1], we have

Proposition 3.1. a) *Given two isometric open embedding of admissible Riemannian pseudomanifolds* $(U,\rho_U)\hookrightarrow(Y_i,\rho_{Y_i})$, $i=1,2$, *and a compact subset* K *of* U, *there is a* $\delta>0$ *such that* $\forall t\in(0,\delta)$ *and* $f^0,\dots,f^k\in C_c^{Lip}(U)$ *with at least one of the* f^j*'s having support inside* K, *one has*

$$\Lambda_k(P(tD_1))|_U(f^0\otimes\cdots\otimes f^k) = \Lambda_k(P(tD_2))|_U(f^0\otimes\cdots\otimes f^k),$$

where D_i *is the signature operator of* (Y_i,ρ_{Y_i}) *as defined in Section 2.1.*

b) *For any* $2q > m = \dim Y$ *and* $f^i\in C^{Lip}(Y)$, *one has*

$$\lim_{t\to 0}\Lambda_{2q}(tD)(f^0\otimes\cdots\otimes f^{2q}) = 0.$$

Moreover, $\Lambda_{2q}(tD) = \partial\int_0^t \nu_{2q+1}(sD)ds$ *where* ν *is defined as in* [MW1, Section 3.2].

c) *The results in* (a) *and* (b) *hold for* $X^\dagger$.

Proof. Same as [MW1, Theorem 2.2, Theorem 3.3]. □

Proposition 3.2. *Let* $\delta_\ell = \frac{1}{2}(1-(-1)^\ell)$. *For any* $f^i\in C^\infty_{SA}(X^\dagger)$,

$$\begin{aligned}&\lim_{t\to 0}\Lambda_{2q}(tD)(f^0\otimes\cdots\otimes f^{2q})\\ =\ &\int_M 2^{\frac{m}{2}}\mathcal{L}(R(g^M))f^0df^1\wedge\cdots\wedge df^{2q}\\ &\qquad -\delta_\ell\,\eta(L)\,2^{\frac{n}{2}}\int_N \mathcal{L}(R(g^N))f^0df^1\wedge\cdots\wedge df^{2q},\end{aligned}$$

where $\mathcal{L}(R(g^M))$ *and* $\mathcal{L}(R(g^N))$ *are the Atiyah-Hirzebruch* $\mathcal{L}$*-polynomials in the curvature of the Levi-Civita connection of the metrics* g^M *and* g^N *respectively.*

Proof. Following [MW2], we choose $\rho_0\in C^\infty([0,1])$ with $\rho_0(r)\in[0,1]$ such that

$$\rho_0(r) = \begin{cases} 1 & \forall r\in[0,\frac{1}{2}],\\ 0 & \forall r\in[\frac{3}{4},1],\end{cases}$$

and define $\rho: X^\dagger\to\mathbb{R}$ by

$$\rho(x) = \begin{cases} 0 & \text{if } x\in M,\\ \rho_0(r) & \text{if } x=(r,s,y)\in c(L)\times N\,.\end{cases}$$

Clearly $\rho\in C^\infty_{SA}(X^\dagger)$.

Let $\mathbf{f} = f^0 \otimes \cdots \otimes f^{2q}$. Write $\mathbf{f} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)}$, where $\mathbf{f}^{(1)} := (\rho f^0) \otimes f^1 \cdots \otimes f^{2q}$ and $\mathbf{f}^{(2)} := ((1-\rho) f^0) \otimes f^1 \cdots \otimes f^{2q}$.

$$\lim_{t\to 0}\langle \Lambda_{2q}(tD), \mathbf{f}\rangle = \lim_{t\to 0}\langle \Lambda_{2q}(tD), \mathbf{f}^{(1)}\rangle + \lim_{t\to 0}\langle \Lambda_{2q}(tD), \mathbf{f}^{(2)}\rangle. \tag{1}$$

By locality (Proposition 3.1) and [MW1, Theorem 3.4],

$$\begin{aligned}\lim_{t\to 0}\langle \Lambda_{2q}(tD), \mathbf{f}^{(2)}\rangle &= 2^{\frac{m}{2}} \int_M \mathcal{L}(R(g^M))(1-\rho) f^0 df^1 \wedge \cdots \wedge df^{2q} \\ &= 2^{\frac{m}{2}} \int_M \mathcal{L}(R(g^M)) f^0 df^1 \wedge \cdots \wedge df^{2q}.\end{aligned}$$

Case 1: ℓ is even and n is odd.

By locality (Proposition 3.1) and product formula [MW2, Proposition 5.1],

$$\lim_{t\to 0}\langle \Lambda_{2q}(tD), \mathbf{f}^{(1)}\rangle = 0.$$

Case 2: ℓ is odd and n is even.

As ℓ is odd, by Thom's Theorem [St, p.183], there exists a smooth, compact and oriented manifold W with $\partial W = 2L$.

Let $Y = (W \cup 2c(L)) \times N$,
$\tilde{M} = 2M \cup (-W \times N)$,
$\pi_Y : Y \to N$ be the projection map onto the second factor,
and $\mathbf{f}_Y = \pi_Y^*(f^0 \otimes \cdots \otimes f^{2q}) \in C^\infty_{SA}(Y)$.

Let $(f)_2$ be the function on $2X$ which equals f on each copy of X. Now $(\mathbf{f}^{(1)})_2 = \mathbf{f}_Y^{(1)} = \mathbf{f}_Y - \mathbf{f}_Y^{(2)}$. By locality (Proposition 3.1),

$$\begin{aligned}2\langle \Lambda_{2q}(tD_X), \mathbf{f}^{(1)}\rangle &= \langle \Lambda_{2q}(tD_{2X}), (\mathbf{f}^{(1)})_2\rangle \\ &= \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y^{(1)}\rangle \\ &= \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y\rangle - \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y^{(2)}\rangle. \qquad (2)\end{aligned}$$

By using the index theorem in [C2] instead of the McKean-Singer formula at the end of the proof in [MW2, Lemma 5.2], we see that we still have a product formula for space with conical singularities. Therefore,

$$\begin{aligned}\langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y\rangle &= sign(W \cup 2c(L)) \cdot \langle \Lambda_{2q}(tD_N), f^0 \otimes \cdots \otimes f^{2q}\rangle \\ &= sign(W \cup 2c(L))\, 2^{\frac{n}{2}} \int_N \mathcal{L}(R(g^N)) f^0 df^1 \cdots df^{2q}. \qquad (3)\end{aligned}$$

The last line follows from [MW1, Theorem 3.4]. Now,

$$2\langle \Lambda_{2q}(tD_X), \mathbf{f}^{(2)}\rangle - \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y^{(2)}\rangle = \langle \Lambda_{2q}(tD_A), \mathbf{f}_A^{(2)}\rangle,$$

where $\mathbf{f}_A^{(2)} = (\mathbf{f}^{(2)})_2 \sqcup \mathbf{f}_Y^{(2)}$ and $A = 2X \cup -Y = \tilde{M} \cup 2(c(L) \cup (-c(L))) \times N$. Let $Z = (c(L) \cup (-c(L))) \times N$. Then $\mathbf{f}_A^{(2)} = \tilde{\mathbf{f}} \sqcup \mathbf{f}_Z^{(2)}$.

Clearly, $\langle \Lambda_{2q}(tD_Z), \mathbf{f}_Z^{(2)}\rangle = 0$. Then by locality (Proposition 3.1),

$$2\langle \Lambda_{2q}(tD_X), \mathbf{f}^{(2)}\rangle - \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y^{(2)}\rangle = \langle \Lambda_{2q}(tD_{\tilde{M}}), \tilde{\mathbf{f}}\rangle. \tag{4}$$

Together with (1), (2), (3) and (4), we have

$$\begin{aligned} & 2\langle \lim_{t\to 0} \Lambda_{2q}(tD), \mathbf{f}\rangle \\ = \; & \lim_{t\to 0} \Big(\langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y\rangle - \langle \Lambda_{2q}(tD_Y), \mathbf{f}_Y^{(2)}\rangle + 2\langle \Lambda_{2q}(tD_X), \mathbf{f}^{(2)}\rangle \Big) \\ = \; & sign(W \cup 2c(L))\ 2^{\frac{n}{2}} \int_N \mathcal{L}(R(g^N)) f^0 df^1 \cdots df^{2q} + \lim_{t\to 0} \langle \Lambda_{2q}(tD_{\tilde{M}}), \tilde{\mathbf{f}}\rangle. \end{aligned}$$

As $\tilde{M}$ is smooth, by [MW1, Theorem 3.4] and the product formula [MW1, Proposition 5.1], we have

$$\begin{aligned} & \lim_{t\to 0} \langle \Lambda_{2q}(tD_{\tilde{M}}), \tilde{\mathbf{f}}\rangle \\ = \; & 2\int_M 2^{\frac{n}{2}} \mathcal{L}(R(g^M)) f^0 df^1 \cdots df^{2q} \\ & \qquad - \int_W 2^{\frac{\ell+1}{2}} \mathcal{L}(R(g^W)) \cdot \int_N 2^{\frac{n}{2}} \mathcal{L}(R(g^N)) f^0 df^1 \cdots df^{2q}. \end{aligned}$$

But by [C2],

$$sign(W \cup 2c(L)) = \int_W 2^{\frac{\ell+1}{2}} \mathcal{L}(R(g^W)) - 2\eta(L).$$

Hence the result follows. ☐

3.2. **Analytic realization of Goresky-MacPherson-Siegel $\mathcal{L}$-class.** In this subsection, we will show that, under a condition on the link, the "straight" Chern character and the Goresky-MacPherson-Siegel $\mathcal{L}$-class are the same (up to constants). This gives an analytic realization of $\mathcal{L}$-class.

In the following, we will denote the $\mathcal{L}$-class as an element in $H_k(X^\dagger)$ and $\overline{H}_k(X^\dagger)$ by the same symbol.

Theorem 3.3. *If the space $X^\dagger$ is even dimensional and $2L$ is oriented cobordant to zero, then*

$$[\ \overline{ch}_{2q}(D)\ ] = [\ 2^q \mathcal{L}_{2q}(X^\dagger)\ ].$$

Proof. Notice that any element in $[X^\dagger, S^{2q}]$ is homotopic to an element in

$$C^\infty_{SA}(X^\dagger, S^{2q}) = \{ f \in C(X^\dagger, S^{2q}) : \text{ smooth on } X \text{ such that } \\ f|_{c(L)\times N} = pr^*(g) \text{ for } g \in C^\infty(N, S^{2q})\}.$$

By [GM], it suffices to show that for any $f \in C^\infty_{SA}(X^\dagger, S^{2q})$ which is transverse,

$$\langle \overline{ch}_{2q}(D), f^*(s^{2q})\rangle \ = 2^q \sigma(f^{-1}(p))\ ,$$

where σ is the signature in Section 1.3 and s^{2q} is a fixed generator of $\overline{H}^k(S^{2q})$. By [BT, p.37], s^{2q} and $f^*(s^{2q})$ are sums of elementary tensors. Note that the normalization constants of $\mathcal{L}$-classes are different from [MS].

Then $f|_{c(L)\times N} = (pr)^*(g)$ for some $g : N \to S^{2q}$ which is smooth and transverse at p. In the rest of this proof, we will adopt the notation used in the proof of Proposition 3.2.

As in the proof of Proposition 3.2, we have

$$\begin{aligned}
& \langle \overline{ch}_{2q}(D), f^*(s^{2q})\rangle \\
= \; & \frac{1}{2}\Big(sign(W\cup 2c(L))\langle \overline{ch}_{2q}(D_N), g^*(s^{2q})\rangle + \langle \overline{ch}_{2q}(D_{\tilde{M}}), \tilde{f}^*(s^{2q})\rangle \Big) \\
= \; & \frac{2^q}{2}\left[\, \sigma(W\cup 2c(L))\,\sigma(g^{-1}(p)) + \sigma\left(2(f^{-1}(p)\cap M)\cup(-W\times g^{-1}(p))\right) \right] \\
& \hfill \text{[MW1], [MS]} \\
= \; & \frac{2^q}{2}\left[\, \sigma((W\cup 2c(L))\times g^{-1}(p)) + \sigma\left(2(f^{-1}(p)\cap M)\cup(-W\times g^{-1}(p))\right) \right] \\
= \; & 2^q\sigma((c(L)\times g^{-1}(p))\cup(f^{-1}(p)\cap M)) \\
= \; & 2^q\sigma(f^{-1}(p)).
\end{aligned}$$

The last three lines follow from the multiplicativity and additivity of σ [Si]. □

Remark 3.4. 1. If the space $X^\dagger$ is even dimensional, then $[\,\mathcal{L}_{2q+1}(X^\dagger)\,] = [\,0\,]$.

2. By Thom's Theorem [St, p.183], $2L$ is oriented cobordant to zero iff all Pontryagin numbers are zero. It is always true if $\ell \not\equiv 0 \mod 4$.

Theorem 3.5. *For any admissible pseudomanifold $X^\dagger$ with one singular stratum such that $2L$ is oriented cobordant to zero, one has*

$$L_*(X^\dagger;\rho_{X^\dagger}) := 2^{2m'}\mathcal{L}_{m'}(R(g^M)) \oplus -\delta_\ell\,\eta(L)\,2^{2n'}\mathcal{L}_{n'}(R(g^N)) \;\in \Omega_*(X^\dagger)_{SA}$$

is a cycle which represents the $\mathcal{L}$-class of $X^\dagger$, where $m' = \frac{m-*}{4}$, $n' = \frac{n-*}{4}, \delta_\ell = \frac{1}{2}(1-(-1)^\ell)$ and $\mathcal{L}_{k'}(R(\cdot)) := 0$ for $k' \notin \mathbb{Z}$.

Proof. We will divide this into two cases.

Case 1: $\dim X$ is even.

It follows from the previous theorem (cf. [MW2, Theorem 4.3]).

Case 2: $\dim X$ is odd.

Let q_1, q_2 be the projections onto the first and second factors of $X^\dagger\times S^1$ respectively. Then by definition,

$$\mathcal{L}_k(X^\dagger)(\omega) \;=\; \mathcal{L}_{k+1}(X^\dagger\times S^1)(\, q_1^*\omega \;\wedge\; q_2^*s^1),$$
$$\text{where} \qquad \omega\in H^k(X^\dagger) \;\text{ and }\; H^1(S^1) = \langle s^1\rangle.$$

The result follows from the definition of the pairing of Sullivan complexes (Section 1.2) and Case 1. □

4. Index theorem

In this section, we will establish the index theorem for the pseudomanifolds with one singular stratum. And then we will identify the "straight" Chern character with the K-homology Chern character.

Theorem 4.1. *If the space $X^\dagger$ is even dimensional, then*

$$Ind(D_E) = 2^{\frac{m}{2}}\int_M \mathcal{L}(R(g^M))\wedge ch(E) \;-\; \delta_\ell\,\eta(L)\,2^{\frac{n}{2}}\int_N \mathcal{L}(R(g^N))\wedge ch(E).$$

Proof. By Proposition 2.9 and Theorem 2.7, we have

$$\int_{c(L)\times N} \mathrm{tr}_s e^{-tD_E^2}(x,x) = \int_{c(L)\times N} \mathrm{tr}_s e^{-t\tilde{D}_E^2}(x,x) \;+\; \mathrm{O}(t)\;\;,$$

where $\tilde{D}_E$ is the twisted signature operator on $c_{0,\infty}(L)\times N$.

Hence, $Ind(D_E) = \lim_{t\to 0}\left(\int_M \mathrm{tr}_s e^{-tD_E^2}(x,x) \; + \; \int_{c(L)\times N} \mathrm{tr}_s e^{-t\tilde{D}_E^2}(x,x)\right).$

1^o ℓ is odd and n is even.

Notice that $\tilde{D}_E = D_{c_{0,\infty}}\hat{\otimes} I + I\hat{\otimes} D_{N,E}$. Therefore, $\mathrm{tr}_s e^{-t\tilde{D}_E^2}(x,x) = \mathrm{tr}_s e^{-tD_{c(L)}^2}(x,x)\ \mathrm{tr}_s e^{-tD_{N,E}^2}(x,x)$. By [C2],

$$\int_{c(L)\times N} \mathrm{tr}_s e^{-t\tilde{D}_E^2}(x,x) = -\eta(L)\int_N 2^{\frac{n}{2}}\mathcal{L}(R(g^N))\wedge ch(E).$$

2^o ℓ is even and n is odd.

Note that $\tilde{D} = D_{c(L)}\otimes I + \phi\otimes D_{N,E}$ where $\phi = (-1)^k$ on $L^2(\Omega^k(c(L)))$. Also, $\gamma = \frac{1}{i}\gamma_1\gamma_2, \gamma_1 = i^{\frac{\ell+2}{2}}c(vol_{c(L)})\otimes I$ and $\gamma_2 = i^{\frac{n+1}{2}}\phi\otimes c(vol_N)$ where $c(vol_{c(L)})$ and $c(vol_N)$ are Clifford multiplication by volume elements in $c(L)$ and N respectively. Then,

$$\begin{aligned}
\mathrm{tr}_s e^{-t\tilde{D}_E^2}(x,x) &= \frac{1}{i}\ \mathrm{tr}(\gamma_1\gamma_2 e^{-tD_{c(L)}^2}\otimes e^{-tD_{N,E}^2})(x,x)\\
&= \frac{1}{i}\ \mathrm{tr}(\gamma_1 e^{-tD_{c(L)}^2}\otimes e^{-tD_{N,E}^2}\gamma_2)(x,x)\\
&= \frac{1}{i}\ \mathrm{tr}(\gamma_2\gamma_1 e^{-tD_{c(L)}^2}\otimes e^{-tD_{N,E}^2})(x,x)\\
&= \frac{1}{i}\ \mathrm{tr}(-\gamma_1\gamma_2 e^{-tD_{c(L)}^2}\otimes e^{-tD_{N,E}^2})(x,x)
\end{aligned}$$

Hence the result follows. □

Remark 4.2. The above theorem is a special case of the index theorem announced in [BC]. They considered the index theory of twisted signature operators for fibration with fibers having conical singularities.

Corollary 4.3. *If the space $X^\dagger$ is even dimensional, then*

$$[\ ch(D)\ ] = [\ \overline{ch}(D)\ ].$$

Proof. By [MS, p.196 and Theorem 9.1], one can see that

$$ch(E_X) = ch(E_M)\oplus ch(E_N) \in H^*(\Omega^*(X^\dagger)_{SA}).$$

Thus, by Theorem 4.1, $Ind(D_E) = \langle 2^{\frac{*}{2}}L_*(X^\dagger;\rho_{X^\dagger}), ch(E)\rangle$. Hence the result follows from Theorem 3.3. □

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