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Abstract. We show that it is consistent with ZFC that every uncountable set can be continuously mapped onto a splitting family.

1. Introduction

A family $\mathcal{A} \subseteq [\omega]^\omega$ is called a splitting family if for every infinite set $B \subseteq \omega$ there exists $A \in \mathcal{A}$ such that

$$|A \cap B| = |(\omega \setminus A) \cap B| = \aleph_0.$$

We denote by $\mathfrak{s}$ the least size of a splitting family. It is well-known that $\aleph_1 \leq \mathfrak{s} \leq 2^{\aleph_0}$. Let

$$\mathsf{S} = \{X \subseteq 2^\omega : \text{ no Borel image of } X \text{ is a splitting family}\}.$$

By "Borel image" we mean image by a Borel function. It is easy to see that S is a σ-ideal containing all countable sets. The purpose of this paper is to show that one cannot prove in ZFC that S contains an uncountable set.

Recall ([5] or [1]) that a forcing notion $(\mathcal{P}, \leq)$ is Suslin if

(1) $\mathcal{P}$ is ccc,
(2) $\mathcal{P}$ is a $\mathbf{\Sigma}^1_1$ set of reals,
(3) relations $\leq, \perp$ are $\mathbf{\Sigma}^1_1$.

Let **MA**(Suslin) denote Martin's Axiom for Suslin partial orders. It is well known that **MA**(Suslin) implies that many cardinal invariants, most notably additivity of measure, are equal to $2^{\aleph_0}$.

Notation used in this paper is standard. In particular, for $s, t \in 2^{<\omega}$, $[s] = \{x \in 2^\omega : x{\restriction}\mathsf{dom}(s) = s\}$ and $s^\frown t$ denotes the concatenation of s and t. For $A, B \subseteq \omega$ define $A \subseteq^\star B$ if $|A \setminus B| < \aleph_0$.

To simplify notation, throughout this paper we will identify elements of $[\omega]^\omega$ with elements of 2^ω via characteristic functions.

2. Consistency result

The goal of this paper is to show **MA**(Suslin) is consistent with $\mathsf{S} = [\mathbb{R}]^{<\aleph_1}$. This is a generalization of a result from [5], where it was proved that **MA**(Suslin)$+\mathfrak{s} = \aleph_1$ is consistent.

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Theorem 2.1. *There exists a model of* $\mathbf{V}' \models \mathsf{ZFC}$ *such that:*

(1) $\mathbf{V}' \models \mathsf{S} = [\mathbb{R}]^{<\aleph_1}$,
(2) $\mathbf{V}' \models \mathbf{MA}(\mathit{Suslin}) + 2^{\aleph_0} = \aleph_2$.

The rest of this section is devoted to the proof of this theorem.

We start with a definition of a forcing notion, due to Hechler (see [4]), that will be crucial for our construction.

Let $\mathsf{Seq}^\star \subseteq \omega^{<\omega}$ be the set of strictly increasing finite sequences. Let

$$\mathbf{D} = \{(s,f) : s \in \mathsf{Seq}^\star,\ s \subseteq f, \text{ and } f \text{ is strictly increasing}\}.$$

For $(s,f),(t,g) \in \mathbf{D}$ define

$$(s,f) \geq (t,g) \iff s \supseteq t \ \&\ \forall n \in \omega\ f(n) \geq g(n).$$

Define a rank function on $\mathbf{D}$:

Definition 2.2 ([2])**.** Suppose that $D \subseteq \mathbf{D}$ is a dense open set. For $s \in \mathsf{Seq}^\star$ define the rank of s as follows:

(1) $\mathsf{rank}_D(s) = 0$ if there exists a function f such that $(s,f) \in D$.
(2) If $\mathsf{rank}_D(s) \neq 0$, then

$$\mathsf{rank}_D(s) = \min\Big\{\alpha : \exists m\ \exists \{s_k : k \in \omega\} \subseteq \mathsf{Seq}^\star \cap \omega^m \\ \big(\mathsf{rank}_D(s_k) < \alpha \ \&\ s \subseteq s_k \ \&\ s_k(|s|) > k\big)\Big\}.$$

Lemma 2.3 ([2]; [1], Lemma 3.5.6)**.** *For every* $s \in \mathsf{Seq}^\star$, $\mathsf{rank}_D(s)$ *is defined.* □

Let $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2 \rangle$ be a finite support iteration such that

(1) $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha$ is Suslin,
(2) if α is a limit ordinal then $\dot{\mathcal{Q}}_\alpha \simeq \mathbf{D}$.

By careful bookkeeping we can ensure that $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathbf{MA}(\text{Suslin}) + 2^{\aleph_0} = \aleph_2$.

We will show that $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathsf{S} = [\mathbb{R}]^{<\aleph_1}$. The following construction is a modification of a construction from [3].

Suppose that $A \subseteq \omega$ is an infinite set. Let A_-, A_+ be two, canonically chosen, disjoint infinite sets such that $A_- \cup A_+ = A$. Assume that $g \in \omega^\omega$ is an increasing function such that $\mathsf{range}(g) \cap A = \{x_n : n \in \omega\}$ is infinite ($x_n < x_{n+1}$ for all n).

Define a real $z_{A,g} \in 2^\omega$ as follows:

$$z_{A,g}(n) = \begin{cases} 1 & \text{if } x_n \in A_+ \\ 0 & \text{if } x_n \in A_- \end{cases} \quad \text{for } n \in \omega.$$

Fix a bijection $\ell : 2^{<\omega} \longrightarrow \omega$ and for $x \in 2^\omega$ define $\ell(x) = \{\ell(x \restriction n) : n \in \omega\}$. Define $S_g : \mathsf{dom}(S_g) \longrightarrow 2^\omega$ as

$$S_g(x) = z_{\ell(x),g} \text{ for } x \in 2^\omega.$$

The following lists some easy properties of the function defined above:

Lemma 2.4. (1) $\mathsf{dom}(S_g)$ *is a* G_δ *subset of* 2^ω,
(2) S_g *is continuous on its domain,*
(3) S_g *extends to a Borel function on* 2^ω.

Proof. (1) Note that $\mathsf{dom}(S_g) = \{x \in 2^\omega : |\mathsf{range}(g) \cap \ell(x)| = \aleph_0\}$, which is a G_δ set (possibly empty). (2) is easy to see and (3) is well-known. □

Definition 2.5. An uncountable set $X \subseteq 2^\omega$ is called a Luzin set if $|F \cap X| \leq \aleph_0$ for every meager set $F \subseteq 2^\omega$.

We have the following easy lemma:

Lemma 2.6. *Every Luzin set is a splitting family.*

Proof. Suppose that $X \subseteq 2^\omega$ is a non-meager set. Since we identify elements of $[\omega]^\omega$ with elements of 2^ω via characteristic functions we can assume that $X \subseteq [\omega]^\omega$. Let $A \in [\omega]^\omega$. Consider the set

$$F = \{z \in [\omega]^\omega : z \subseteq^\star (\omega \setminus A) \text{ or } A \subseteq^\star z\}.$$

It is easy to see that F is a meager set, and that any element of $X \setminus F$ splits A. $\square$

Lemma 2.7. *Suppose that d is a $\mathbf{D}$-generic real over $\mathbf{V}$. If $Z \subseteq 2^\omega \cap \mathbf{V}$ is uncountable then $S_d(Z)$ is a Luzin set in $\mathbf{V}[d]$.*

Proof. Observe first that by genericity $\mathbf{V} \cap 2^\omega \subseteq \mathsf{dom}(S_d)$ and S_d is one-to-one on $\mathbf{V} \cap 2^\omega$. In particular, $S_d(Z)$ is an uncountable set.

Suppose that $F \in \mathbf{V}[d]$ is a closed nowhere dense subset of 2^ω. To show that $S_d(Z)$ is a Luzin set it is enough to show that $S_d(Z) \cap F$ is countable. Let $f \in (2^{<\omega})^\omega \cap \mathbf{V}[d]$ be a function defined as follows:

$$f(n) = \min\{s \in 2^{<\omega} : \forall t \in 2^{\leq n} \ [t^\frown s] \cap F = \emptyset\}.$$

(The minimum is taken with respect to some canonical enumeration of $2^{<\omega}$.) It is well-known that such an s exists.

Let $\dot{f}$ be a $\mathbf{D}$-name for f and define for $n \in \omega$,

$$D_n = \{p \in \mathbf{D} : \exists s \in 2^{<\omega} \ p \Vdash_{\mathbf{D}} \dot{f}(n) = s\}.$$

Let $N \prec \mathbf{H}(\lambda)$ be a countable model containing $\dot{f}$ and Z, where λ is a sufficiently large regular cardinal.

Lemma 2.8. *If $x \in \mathbf{V} \cap 2^\omega$ but $x \notin N \cap 2^\omega$ then $S_d(x) \notin F$.*

Proof. Suppose not and let $x \notin N \cap 2^\omega$ be a counterexample. Choose $(s, g) \in \mathbf{D}$ such that

$$(s, g) \Vdash_{\mathbf{D}} S_{\dot{d}}(x) \in \dot{F}.$$

Let $\widetilde{k} = |\mathsf{range}(s) \cap \ell(x)|$. In other words, $S_d(x) \restriction \widetilde{k}$ is determined by (s, g). Let

$$U = \{t \in \mathsf{Seq}^\star : s \subseteq t \ \& \ |\mathsf{range}(t) \cap \ell(x)| = \widetilde{k} \ \& \ \forall j \in \mathsf{dom}(t) \setminus \mathsf{dom}(s) \ t(j) \geq g(j)\}.$$

Lemma 2.9. $\min\{\mathsf{rank}_{D_{\tilde{k}}}(t) : t \in U\} = 0.$

Proof. Suppose that the lemma is not true and let $t \in U$ be an element of minimal rank. By the definition there exists m and a sequence $\{t_j : j \in \omega\} \subseteq \mathsf{Seq}^\star \cap \omega^m$ such that for every j:

(1) $t \subseteq t_j$,
(2) $\mathsf{rank}_{D_{\tilde{k}}}(t_j) < \mathsf{rank}_{D_{\tilde{k}}}(t)$,
(3) $t_j(|t|) > j$.

Fix i such that $|t| \leq i < m$ and let $W_i = \{t_j(i) : j \in \omega\}$. Note that every subsequence of $\{t_j : k \in \omega\}$ witnesses that $\mathsf{rank}_{D_{\tilde{k}}}(t) > 0$ as well. Thus, by passing to a subsequence we can assume that there is a set $\ell(x_i)$ such that $W_i \subseteq \ell(x_i)$ or $W_i \cap \ell(x)$ is finite for all x. In particular, if such a real x_i exists it is a member of N.

Since $x \notin N$, $W_i \cap \ell(x)$ is finite for all $|t| \leq i < m$. Therefore, there exists j such that $\mathsf{range}(t_j) \cap \ell(x) = \mathsf{range}(t) \cap \ell(x)$. In particular, $t_j \in U$ and $\mathsf{rank}_{D_{\tilde{k}}}(t_j) < \mathsf{rank}_{D_{\tilde{k}}}(t)$, which is a contradiction. □

Let $t \in U$ be such that $\mathsf{rank}_{D_{\tilde{k}}}(t) = 0$. There exists $h \in \omega^\omega$ such that $(t, h) \in D_{\tilde{k}}$. Therefore, $(t, \max(h, g)) \geq (s, g)$ and $(t, \max(h, g))$ decides the value of $\dot{f}(\widetilde{k})$. Denote this value by $\widetilde{s}$. However, $(t, \max(h, g))$ does not put any restrictions on values of $S_d(x)(j)$ for $j \geq \widetilde{k}$. Extend t to t' such that

$$(t', \max(h, g)) \Vdash_{\mathbf{D}} \left(S_{\dot{d}}(x){\restriction}\widetilde{k}\right)^\frown \widetilde{s} \subseteq S_{\dot{d}}(x).$$

It is clear that

$$(t', \max(h, g)) \Vdash_{\mathbf{D}} S_{\dot{d}}(x) \notin \dot{F}.$$

This contradiction ends the proof of Lemma 2.7. □

Let $G \subseteq \mathcal{P}_{\omega_2}$ be a generic filter over $\mathbf{V}$. Suppose that $Z \subseteq 2^\omega \cap \mathbf{V}[G]$ is a set of cardinality $\aleph_1$. First we find a limit ordinal α such that $Z \subseteq \mathbf{V}[G \cap \mathcal{P}_\alpha]$. We will work in the model $\mathbf{V}_1 = \mathbf{V}[G \cap \mathcal{P}_{\alpha+1}] = \mathbf{V}[G \cap \mathcal{P}_\alpha][d]$, where d is a $\mathbf{D}$-generic real over $\mathbf{V}[G \cap \mathcal{P}_\alpha]$.

To finish the proof it is enough to show that $S_d(Z)$ is a splitting family in $\mathbf{V}[G]$. Note however that $S_d(Z)$ is not a Luzin set in $\mathbf{V}[G]$. In fact, $S_d(Z)$ is meager in $\mathbf{V}[G]$.

Lemma 2.10. $\{S_d(x) : x \in Z\}$ *is a splitting family in* $\mathbf{V}[G]$.

Proof. We will work in $\mathbf{V}_1$. By 2.7, we know that $S_d(Z)$ is a Luzin set in in $\mathbf{V}_1$. Note that $\mathbf{V}[G]$ is a generic extension of $\mathbf{V}_1$ via finite support iteration of Suslin forcings $\mathcal{P}_{\alpha+1,\omega_2}$.

Let $\dot{A}$ be a $\mathcal{P}_{\alpha+1,\omega_2}$-name for a set $A \in [\omega]^\omega$. We will need the following lemma:

Lemma 2.11. *For every* $p \in \mathcal{P}_{\alpha+1,\omega_2}$, *the set*

$$Z_p = \left\{z \in Z : p \Vdash_{\alpha+1,\omega_2} \text{“}S_d(z) \subseteq^\star (\omega \setminus \dot{A}) \text{ or } \dot{A} \subseteq^\star S_d(z)\text{”}\right\}$$

is countable.

Before we prove the lemma notice that the theorem follows from it immediately – given $p \in \mathcal{P}_{\alpha+1,\omega_2}$, $\dot{A}$ and $z \in Z \setminus Z_p$ we can find $q \geq p$ such that $q \Vdash$ “$S_d(z)$ splits $\dot{A}$.”

Proof of the lemma. We will use the absoluteness properties of Suslin forcing (see [1] or [5]).

Fix a condition $p \in \mathcal{P}_{\alpha+1,\omega_2}$. Let M be a countable elementary submodel of $\mathbf{H}(\lambda)$ containing $\dot{A}$, p and $\mathcal{P}_{\alpha+1,\omega_2}$. Define a finite support iteration $\langle \mathcal{P}_\alpha(M), \dot{\mathcal{Q}}_\alpha(M) : \alpha < \omega_2\rangle$ as follows:

$$\Vdash_\alpha \dot{\mathcal{Q}}_\alpha(M) = \begin{cases} \dot{\mathcal{Q}}_\alpha & \text{if } \alpha \in M \\ \emptyset & \text{if } \alpha \notin M \end{cases} \quad \text{for } \alpha < \omega_2.$$

Let $\mathcal{P} = \lim \mathcal{P}_\alpha(M)$. $\mathcal{P}$ is the part of the iteration that contains all information regarding $\dot{A}$. $\mathcal{P}$ is isomorphic to a countable iteration of Suslin forcings. In particular, $\mathcal{P}$ has a definition that can be coded as a real number (essentially by encoding M as a real number).

From Suslinness it follows that $\mathcal{P} \lessdot \mathcal{P}_{\alpha+1,\omega_2}$ and that $\dot{A}$ is a $\mathcal{P}$-name (see [1], Lemma 9.7.4, or [5]). Moreover, it is enough to show that

$$\Big\{z \in Z : p \Vdash_{\mathcal{P}} \text{“}S_d(z) \subseteq^\star (\omega \setminus \dot{A}) \text{ or } \dot{A} \subseteq^\star S_d(z)\text{”}\Big\}$$

is countable.

Let $N \prec \mathbf{H}(\lambda)$ be a countable model containing $M, \dot{A}$ and $\mathcal{P}$. Since $S_d(Z)$ is a Luzin set in $\mathbf{V}_1$, the set

$$Z_0 = \{z \in Z : S_d(z) \text{ is not a Cohen real over } N\}$$

is countable. We will show that $Z_p \subseteq Z_0$. In particular, for $z \in Z \setminus Z_0$,

$$p \not\Vdash_{\mathcal{P}} \text{“}S_d(z) \subseteq^\star (\omega \setminus \dot{A}) \;\text{ or }\; \dot{A} \subseteq^\star S_d(z)\text{”},$$

which will finish the proof. Fix $z \in Z \setminus Z_0$ and let $Y = S_d(z)$ be a Cohen real over N. Without loss of generality we can assume that $p \Vdash_{\mathcal{P}} Y \subseteq^\star (\omega \setminus \dot{A})$.

Clearly, $N[Y][G \cap N[Y]] \models Y \subseteq^\star (\omega \setminus \dot{A}[G \cap N[Y]])$ and therefore

$$N[Y] \models \text{“}p \Vdash_{\mathcal{P}} Y \subseteq^\star (\omega \setminus \dot{A}),\text{”}$$

since the last statement is absolute. Represent the Cohen algebra as $\mathbf{C} = [\omega]^{<\omega}$ and let $\dot{Y}$ be the canonical name for a Cohen real. There is a condition $s \in \mathbf{C}$ such that

$$N \models s \Vdash_{\mathbf{C}} \text{“}p \Vdash_{\mathcal{P}} \dot{Y} \subseteq^\star (\omega \setminus \dot{A}).\text{”}$$

Let $Y' = s \cup \Big(\omega \setminus \big(Y \setminus \max(s)\big)\Big)$. Y' is also a a Cohen real over N and since $s \subseteq Y'$ we get that $N[Y'] \models$ “$p \Vdash_{\mathcal{P}} Y' \subseteq^\star (\omega \setminus \dot{A})$.” It follows that

$$N[Y'][G \cap N[Y']] \models Y' \subseteq^\star (\omega \setminus \dot{A}[G \cap N[Y']]).$$

Note that $\dot{A}[G] = \dot{A}[G \cap N[Y']] = \dot{A}[G \cap N[Y]]$. Thus $\mathbf{V}[G] \models Y \cup Y' \subseteq^\star (\omega \setminus \dot{A}[G])$ which means that $\dot{A}[G]$ is finite. Contradiction.

The same argument shows that the assumption that $p \Vdash_{\mathcal{P}} \dot{A} \subseteq^\star Y$ leads to a contradiction. □

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References

[1] Tomek Bartoszyński and Haim Judah, *Set Theory: on the structure of the real line*, A.K. Peters, 1995. CMP 96:01

[2] James E. Baumgartner and Peter Dordal, *Adjoining dominating functions*, The Journal of Symbolic Logic **50** (1985), no. 1, 94–101. MR **86i:**03064

[3] Jorg Brendle, Haim Judah, and Saharon Shelah, *Combinatorial properties of Hechler forcing*, Annals of Pure and Applied Logic **59** (1992), 185–199. MR **93k:**03048

[4] S. H. Hechler, *On the existence of certain cofinal subsets of* ω^ω, Axiomatic Set Theory (T. J. Jech, ed.), Proc. Symp. Pure Math., vol. 13, Amer. Math. Soc., Providence, R.I., 1974, Part 2, pp. 155–173. MR **50:**12716

[5] Haim Judah and Saharon Shelah, *Suslin forcing*, The Journal of Symbolic Logic **53** (1988), 1188–1207. MR **90h:**03035

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