

ON BERNSTEIN-SATO POLYNOMIALS

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ABSTRACT. We show that for fixed n and d the set of Bernstein-Sato polynomials of all the polynomials in at most n variables of degrees at most d is finite. As a corollary, we show that there exists an integer t depending only on n and d such that f^{-t} generates R_f as a module over the ring of the k -linear differential operators of R , where k is an arbitrary field of characteristic 0, R is the ring of polynomials in n variables over k and $f \in R$ is an arbitrary non-zero polynomial of degree at most d .

Throughout this paper k is a field of characteristic 0, $R_n(k) = k[X_1, \dots, X_n]$ is the ring of polynomials in v variables over k and $A_n(k)$ is the corresponding Weyl algebra, i.e. the subring of $\text{Hom}_k(R_n(k), R_n(k))$ generated by the multiplications by all the elements of $R_n(k)$ and the n partial differentiations $d_1 = \frac{d}{dX_1}, \dots, d_n = \frac{d}{dX_n}$. Every element $Q \in A_n(k)$ may be uniquely written as a finite sum

$$Q = \sum \sigma_{i_1, \dots, i_n, j_1, \dots, j_n} X_1^{i_1} \dots X_n^{i_n} d_1^{j_1} \dots d_n^{j_n}$$

where all $i_1, \dots, i_n, j_1, \dots, j_n$ are non-negative integers and $\sigma_{i_1, \dots, i_n, j_1, \dots, j_n} \in k$. Elementary facts about $A_n(k)$ will be used below without specific references; they all may be found, for example, in [B, pp. 1–15].

For every polynomial $f \in R_n(k)$ there exists a functional equation

$$(*) \quad b(s)f^s = \sum_{j=0}^{j=v} s^j Q_j f^{s+1},$$

where s is a new variable, $b(s) \in k[s]$ is a polynomial in s and $Q_0, \dots, Q_v \in A_n(k)$. The polynomials $b(s)$ for which equation (*) exists form an ideal in $k[s]$, and the monic generator of this ideal is called the Bernstein-Sato polynomial of f and denoted $b_f(s)$. The roots of $b_f(s)$ are rational numbers; in particular, $b(s) \in \mathbb{Q}[s]$, where \mathbb{Q} is the field of rational numbers.

The natural $A_n(k)$ -module structure on $R_n(k)$ induces a structure of $A_n(k)$ -module on $R_n(k)_f$ and this $A_n(k)$ -module is known to be cyclic, so it is generated by f^{-t} for some $t \geq 0$. We denote the smallest t with this property by t_f .

The simplest numerical characteristics of a polynomial $f \in R_n(k)$ are its degree d and its number of variables (i.e. n). This short note is motivated by the following question: What can one say about $b_f(s)$ and t_f in terms of d and n ?

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Proposition-Definition. Let $B(n, d)$ be the set of all the polynomials $b(s) \in \mathbb{Q}[s]$ such that $b(s) = b_f(s)$ for some polynomial f of degree at most d in at most n variables with coefficients in some field of characteristic 0. Then $B(n, d)$ is a finite set for all n and d .

Corollary-Definition. For all n and d there exists an integer t such that $t_f \leq t$ for all polynomials f of degree at most d in at most n variables and with coefficients in any field of characteristic 0. We denote the smallest such integer by $t(n, d)$.

Proof of the corollary. One can set t to be the biggest absolute value of all the negative integer roots of all the elements of the finite set $B(n, d)$. For then $b_f(-t') \neq 0$ for every integer $t' > t$, so equation (*) with $b(s) = b_f(s)$ shows upon dividing by $b_f(-t') \in \mathbb{Q}^*$ that $f^{-t'}$ belongs to the $A_n(k)$ -submodule generated by $f^{-(t'-1)}$, so by induction on $t' - t$ it follows that $f^{-t'}$ belongs to the $A_n(k)$ -submodule generated by f^{-t} , i.e. f^{-t} generates all of $R_n(k)_f$. \square

Proof of the proposition. Let F be a polynomial in n variables X_1, \dots, X_n of degree d and with undetermined coefficients. Let $B_k = k[\text{coefficients of } F]$ be the polynomial ring over k in the coefficients of F . Let $N = N(n, d)$ be the number of monomials in n variables of degree $\leq d$. Then B_k is the coordinate ring of \mathbb{A}_k^N , the affine N -space over k . The set of polynomials in n variables of degree at most d and with coefficients from k is in a one-to-one correspondence with the set of k -rational points of \mathbb{A}_k^N .

Let $Y \subset \mathbb{A}_{\mathbb{Q}}^N$ be a closed reduced irreducible subscheme, where \mathbb{Q} is the field of rational numbers. Let $P \subset B_{\mathbb{Q}}$ be the defining prime ideal of Y , let $\mathbb{Q}[Y] = B_{\mathbb{Q}}/P$ be the coordinate ring of Y and let $\mathbb{Q}(Y)$ be the fraction field of $\mathbb{Q}[Y]$. The natural surjection $B_{\mathbb{Q}} \rightarrow B_{\mathbb{Q}}/P = \mathbb{Q}[Y]$ followed by the natural inclusion $\mathbb{Q}[Y] \rightarrow \mathbb{Q}(Y)$ produces a map $B_{\mathbb{Q}} \rightarrow \mathbb{Q}(Y)$. Let $g \in \mathbb{Q}(Y)[X_1, \dots, X_n]$ be the polynomial whose coefficients are the images of the corresponding coefficients of F under this map $B_{\mathbb{Q}} \rightarrow \mathbb{Q}(Y)$. Then g satisfies a functional equation

$$b_g(s)g^s = \sum_{j=0}^{j=v} s^j Q_j g^{s+1}$$

with all $Q_j \in A_n(\mathbb{Q}(Y))$. All the coefficients $\sigma_{i_1, \dots, i_n, j_1, \dots, j_n}$ of all the Q_j and all the coefficients of g are elements of $\mathbb{Q}(Y)$, hence quotients of two elements of $\mathbb{Q}[Y]$. Since there are only finitely many non-zero coefficients, they all have a common denominator, which we call c . Thus the coefficients of g and of all the Q_j are in $\mathbb{Q}[Y]_c$.

Let $U(c) \subset Y$ be the open subscheme whose coordinate ring is $\mathbb{Q}[Y]_c$. Let f be a polynomial with coefficients in a field k that corresponds to a k -rational point lying on $U(c) \otimes_{\mathbb{Q}} k \subset \mathbb{A}_{\mathbb{Q}}^N \otimes_{\mathbb{Q}} k = \mathbb{A}_k^N$. This k -rational point corresponds to a \mathbb{Q} -algebra homomorphism $\phi : \mathbb{Q}[Y]_c \rightarrow k$ that sends the coefficients of g to the corresponding coefficients of f . Applying ϕ to all the coefficients $\sigma_{i_1, \dots, i_n, j_1, \dots, j_n}$ of Q_j from the above functional equation we get $Q'_j \in A_n(k)$. Since $b_g(s) \in \mathbb{Q}[s]$, applying ϕ to the coefficients of $b_g(s)$ we get $b_g(s)$ again. Thus applying ϕ to all the coefficients of the above functional equation we get the functional equation

$$b_g(s)f^s = \sum_{j=0}^{j=v} s^j Q'_j f^{s+1}.$$

Hence the set of the Bernstein-Sato polynomials of all polynomials f that correspond to k -rational points lying on $U(c) \otimes_{\mathbb{Q}} k$ is a subset of the set of the monic divisors of $b_g(s)$. Since $b_g(s)$ depends only on g , but not on f , this set is finite.

We are going to prove the following claim by induction on $\delta = \dim Y$: the set of polynomials $b(s)$ such that $b(s) = b_f(s)$ for some polynomial f that corresponds to a k -rational point lying on $Y \otimes_{\mathbb{Q}} k \subset \mathbb{A}_k^n$, where k is some field extension of \mathbb{Q} , is finite. The proposition then follows by putting $Y = \mathbb{A}_{\mathbb{Q}}^N$.

If $\delta = 0$, then Y is a closed point, so $\mathbb{Q}[Y]$ is a field, so we may take $c = 1$, i.e. $U(c) = Y$. As is shown above, the set of the Bernstein-Sato polynomials of all the polynomials f that correspond to points on $U(c) \otimes_{\mathbb{Q}} k$ is finite. This finishes the case $\delta = 0$. If $\delta > 0$, then f corresponds either to a point on $U_c \otimes_{\mathbb{Q}} k$, or to a point on $(Y \setminus U(c)) \otimes_{\mathbb{Q}} k$. If f corresponds to a point on $U(c) \otimes_{\mathbb{Q}} k$, it has been shown above that the set of possible candidates for the Bernstein-Sato polynomial of f is finite. If f corresponds to a point on $(Y \setminus U(c)) \otimes_{\mathbb{Q}} k$, then, since $Y \setminus U(c)$ is a union of a finite number of closed irreducible subschemes of dimension $< \delta$, we are done by induction on δ . \square

Question. Is the subset of \mathbb{A}_k^N that corresponds to a fixed Bernstein-Sato polynomial constructible?

Remarks on computability. (a) Given f , it is not hard to describe an algorithm for finding equation (*). Indeed, for a fixed d , to check whether such an equation exists with $v = \deg b(s) = \deg Q_j = d$, reduces to checking whether a corresponding system of linear equations over k whose unknowns are the coefficients of $b(s), Q_0, \dots, Q_d$ is consistent. Here the degree of an element $Q \in A_n(k)$ is the biggest δ such that Q has a non-zero coefficient $\sigma_{i_1, \dots, i_n, j_1, \dots, j_n}$ with $i_1 + \dots + i_n + j_1 + \dots + j_n = \delta$. Since (*) exists for some d , by checking the consistency of this system for $d = 1, 2, 3, \dots$ one will eventually find a d for which it is consistent and its solution provides a desired equation (*).

(b) To compute $b_f(s)$ for a given f is a very non-trivial problem, and at present there does not seem to be known an algorithm for this (cf. [G, p. 419]). Of course, the algorithm described in (a) above produces a polynomial $b(s)$ which is a multiple of $b_f(s)$, hence it produces a finite number of possible candidates for $b_f(s)$, but there does not seem to be known an algorithm to decide whether a given polynomial equals $b_f(s)$, so one does not know how to pick the right polynomial from the finite set of candidates.

(c) An algorithm for computing t_f does not seem to be known at present. However, the algorithm described in (a) above enables one to compute an upper bound on t_f . Namely, if $b(s)$ is a polynomial for which (*) exists, then the biggest absolute value of its negative integer roots is an upper bound on t_f , as is clear from the proof of the corollary.

(d) Considering that there is an algorithm to produce (*), the procedure from the proof of the proposition is computable (while there is no algorithm for producing $b_g(s)$, there is an algorithm for producing some polynomial $b(s) \in \mathbb{Q}(Y)[s]$ that fits a functional equation for g , and if $\tilde{b}(s)$ is the greatest divisor of $b(s)$ having rational roots, then $b_f(s)$ is a monic divisor of $\tilde{b}(s)$). This provides an algorithm for producing a finite set of polynomials from $\mathbb{Q}[s]$ containing $B(n, d)$. The biggest absolute value of all the negative integer roots of all those polynomials provides an upper bound on $t(n, d)$. But there does not seem to be known an algorithm for computing $B(n, d)$, nor for $t(n, d)$.

(e) It would certainly be very interesting to have not just an algorithm, but an explicit upper bound on $t(n, d)$, let alone an explicit formula for $t(n, d)$.

ADDED IN PROOF

Very recently Toshinori Oaku found an algorithm for the computation of $b_f(s)$ for an arbitrary polynomial f .

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