

## $C^k$ CONJUGACY OF 1-D DIFFEOMORPHISMS WITH PERIODIC POINTS

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(Communicated by Mary Rees)

ABSTRACT. It is shown that the set of heteroclinic orbits between two periodic orbits of saddle-node type induces functional moduli which are completely contained in a new ‘transition map’. For one-dimensional  $C^2$  diffeomorphisms with saddle-node periodic points, two such diffeomorphisms are  $C^2$  conjugated if and only if the transition maps of their heteroclinic orbits are the same. An equivalent transition map is given for  $C^k$  diffeomorphisms with hyperbolic periodic points, and it is shown that this transition map is an invariant of  $C^k$  conjugation. However, in this case the transition map alone is sufficient to guarantee conjugacy only in a limited sense.

### 1. INTRODUCTION

Two  $C^k$  diffeomorphisms  $f$  and  $g$  are  $C^k$  conjugated if there exists a  $C^k$  diffeomorphism  $\phi$  such that

$$(1) \quad \phi \circ f = g \circ \phi.$$

If  $\phi$  is merely a homeomorphism we say  $f$  and  $g$  are topologically conjugated. Smooth conjugacy of diffeomorphisms of the circle without periodic points has been studied extensively and many deep results have been found (see [MS] for an overview and references).

If two diffeomorphisms are conjugated, then they must be locally conjugated in neighborhoods of their respective periodic points. It is well known that a  $C^k$  diffeomorphism is  $C^{k-1}$  conjugate in a neighborhood of a hyperbolic fixed point to the linearization of the map at the fixed point if certain nonresonance conditions on the eigenvalues of the linear map are satisfied. For a one dimensional  $C^k$  diffeomorphism with  $k \geq 2$ , these conditions are satisfied vacuously and it is known that the conjugation is  $C^k$ . It is less well-known that all  $C^2$  diffeomorphisms at quadratic saddle-node fixed points are  $C^1$  conjugate to the simple map  $x \mapsto x + x^2$  [KCG]. Analogous results hold if  $f$  has periodic points. Thus a necessary condition for global conjugation of two diffeomorphisms in the class  $C^2$  is that there is a correspondence of the periodic points of the two maps and agreement of the derivatives of the maps at the corresponding periodic points.

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Received by the editors June 14, 1995 and, in revised form, January 9, 1996.

1991 *Mathematics Subject Classification*. Primary 34C35, 58C25.

*Key words and phrases*. Smooth conjugacy, global conjugacy.

The author was partially supported by AFOSR grant #F49620-93-1-0147.

Conjugacy of diffeomorphisms with hyperbolic fixed points was studied by Belitski [Be]. He showed that conjugacy of these maps depends on functional moduli which are  $C^{k-1}$  positive valued functions on the circle (see also [IY] and [KH]). Among other things Belitski obtained that if two  $C^k$  maps are  $C^1$  conjugate, then they are also  $C^{k-1}$  conjugate. In the present paper we introduce a new moduli which seems to be more constructive than Belitski's. The new moduli, which are contained in what we call a 'transition map', allow for nonhyperbolic fixed points, whereas hyperbolicity is essential to Belitski's definition. It is interesting that the transition map completely determines conjugacy for maps with nonhyperbolic periodic points, but they determine conjugacy in the case of hyperbolic periodic points only in a limited sense.

The approach of this paper is related to conventional multipliers of homoclinic orbits, a new tool in the theory of nonlocal bifurcations. The multiplier was used in [ACL] to study higher dimensional bifurcations which result in the appearance of Lorenz-type attractors and in [ALY] to study behavior of a circle map inside the intervals of phase locking. It was noted in [ALY] that the multipliers are an invariant of  $C^1$  conjugation near a saddle-node fixed point with a homoclinic loop, giving motivation for the present work.

## 2. TRANSITION MAPS OF HETEROCLINIC ORBITS OF SADDLE-NODE POINTS

To begin, we suppose that all of the functions under consideration are  $C^2$  diffeomorphisms, and have  $n$  quadratic saddle-node fixed points. The collection of such maps we denote by  $SN_n^2(X)$ , where  $X$  denotes either  $\mathbf{R}$  or  $\mathbf{S}^1$ . If  $f$  and  $g$  are both in  $SN_n^2(X)$ , then they are topologically conjugate.

Suppose that  $\bar{x}_j$  and  $\bar{x}_{j+1}$  are adjacent saddle-node fixed points of  $f$ . Choose  $a_0 \in (\bar{x}_j, \bar{x}_{j+1})$  and let  $\{a_i\}_{i=-\infty}^{\infty}$  be the sequence defined by  $a_{i+1} = f(a_i)$ , and let  $I_i = [a_i, a_{i+1})$ . We will call  $a_0$  the **base point** and the intervals  $I_i$  the **fundamental intervals**. Denote by  $\Gamma_j$  the set of orbits of  $f$  in  $(\bar{x}_j, \bar{x}_{j+1})$ , and by  $[x_0]$  the orbit through a point  $x_0$ . Note that  $\Gamma_j$  intersected with any  $I_i$  contains exactly one point. Then given  $x_0 \in I_0$ , let  $x_i = [x_0] \cap I_i = f^i(x_0)$  and let  $\xi_i$  be the ratio  $(x_i - a_i)/(a_{i+1} - a_i)$ . Consider the limits

$$(2) \quad u^\pm([x_0], a_0) = \lim_{i \rightarrow \pm\infty} \xi_i = \lim_{i \rightarrow \pm\infty} \frac{x_i - a_i}{a_{i+1} - a_i}.$$

**Lemma 1** ([ALY]). *The limits  $u^\pm([x_0])$  exist and are independent of the choice of local coordinates. If  $u^\pm(x_0)$  and  $\bar{u}^\pm(x_0)$  are defined as above for different choices of base points  $a_0$  and  $\bar{a}_0$  respectively,  $a_0 < \bar{a}_0$ , then  $u^\pm([x_0]) = \bar{u}^\pm([x_0]) + u^\pm([\bar{a}_0])$ , if  $x_0 > \bar{a}_0$ ; and  $u^\pm([x_0]) = \bar{u}^\pm([x_0]) - 1 + u^\pm([\bar{a}_0])$ , if  $x_0 < \bar{a}_0$ . Furthermore, the sequence  $\{\xi_i(\cdot)\}$  converges in the  $C^2$  topology on the interval  $I_0$ .*

Note that the limit  $u^+$  associates with each orbit a point on  $[0, 1)$ . We may consider  $u^\pm$  as maps from  $\Gamma_j$  to the unit circle  $S^1$ , which takes  $[a_0]$  to a marked point 0. The limit  $u_j^+ : (\Gamma_j, [a_0]) \rightarrow (\mathbf{S}^1, 0)$  we call the **forward coordinates** of the orbits in  $(\bar{x}_j, \bar{x}_{j+1})$  and  $u_j^- : (\Gamma_j, [a_0]) \rightarrow (\mathbf{S}^1, 0)$  we call the **backward coordinates** of the orbits. The map  $B_j \equiv u_j^+ \circ (u_j^-)^{-1} : (\mathbf{S}^1, 0) \rightarrow (\mathbf{S}^1, 0)$ , that takes the backward coordinate of an orbit to the forward coordinate, we call the **transition map** of the orbits between  $\bar{x}_j$  and  $\bar{x}_{j+1}$ , and we call  $\mathcal{B}_f = \{B_1, \dots, B_n\}$  the **moduli** of  $f$ . We note that  $\mathcal{B}_f$  is unique except for choice of the ordering of the fixed points and the choice of base points on the connecting intervals.

**Corollary 1.** *Suppose  $B_j$  and  $\bar{B}_j$  are transition maps for a single function  $f$  on an interval  $(\bar{x}_j, \bar{x}_{j+1})$  with respect to base points  $a_0$  and  $\bar{a}_0$  respectively.*

$$B_j = R_{\bar{B}_j(u^-(\bar{a}_0))} \circ \bar{B}_j \circ R_{-u^-(\bar{a}_0)},$$

where  $R_\theta$  denotes a rigid rotation by an angle  $\theta$ .

We will call the moduli of two functions **equivalent**,  $B_f \sim B_g$ , if there exist: 1a) for  $\mathbf{S}^1$ , a cyclic permutation of the fixed points, or, 1b) for  $\mathbf{R}$ , a reversal of order of the fixed points; and, 2) angles  $\{\theta_1, \dots, \theta_n\}$  such that

$$(3) \quad B_{f,j} = R_{B_{g,j}(\theta)} \circ B_{g,j} \circ R_{-\theta_j}.$$

for each  $j$ , where  $R_\theta$  denotes a rotation by an angle  $\theta$ . The main result follows.

**Theorem 1.** *Maps  $f$  and  $g$  in  $SN_n^2(X)$  are  $C^2$  conjugate if and only if  $B_f \sim B_g$ .*

*Proof.* Suppose that  $\phi \circ f = g \circ \phi$ ; then  $\phi \circ f^i = g^i \circ \phi$ . For  $a_0$  and  $x_0$  in the domain of  $f$ , let  $\bar{a}_0 = \phi(a_0)$  and  $\bar{x}_0 = \phi(x_0)$ . Let  $\bar{\xi}_i$  denote the ratio (2) for  $\bar{x}_0$ , i.e.

$$\bar{\xi}_i = \frac{\phi(x_i) - \phi(a_i)}{\phi(a_{i+1}) - \phi(a_i)}.$$

Let  $\bar{x}_{j+1}$  have local coordinate 0. Since  $\phi$  is a diffeomorphism

$$\begin{aligned} \bar{\xi}_i &= \frac{\phi'(0)(x_i - a_i) + o(a_i)}{\phi'(0)(a_{i+1} - a_i) + o(a_i)} \\ &= \frac{x_i - a_i}{a_{i+1} - a_i} + o(a_i). \end{aligned}$$

Thus,  $\bar{\xi}_i = \xi_i + o(a_i)$  as  $i \rightarrow \infty$ . The two limits are therefore the same. Since they must be the same in the backward direction as well, the transition maps are the same.

Now we assume that the transition map of  $f$  for the heteroclinic orbit between two fixed points  $\{\bar{x}_j, \bar{x}_{j+1}\}$  is equivalent to the transition map of  $g$  between two fixed points  $\{\bar{y}_j, \bar{y}_{j+1}\}$ . By Cor. 1, we may rechoose base points  $a_0$  and  $b_0$  so that  $B_{f,j} = B_{g,j}$ . We define a smooth map from  $(\bar{x}_j, \bar{x}_{j+1})$  to  $(\bar{y}_j, \bar{y}_{j+1})$  in the following way. Fix  $a_0$  and let  $\tilde{u}^+ : (\bar{x}_j, \bar{x}_{j+1}) \rightarrow \mathbf{R}$  be the map given by  $\tilde{u}^+(x_i) = u^+(x_0) + i$ . In a similar way define  $\tilde{v}^+ : (\bar{y}_j, \bar{y}_{j+1}) \rightarrow \mathbf{R}$ . Let  $\phi$  be defined on each interval  $(\bar{x}_j, \bar{x}_{j+1})$  to be the function

$$(4) \quad \phi \equiv (\tilde{v}^+)^{-1} \circ \tilde{u}^+.$$

Because  $B_{f,j} = B_{g,j}$  we also have

$$\phi = (\tilde{v}^-)^{-1} \circ \tilde{u}^-.$$

It is clear that for this choice of  $\phi$ , equation (1) holds on the interval  $(\bar{x}_j, \bar{x}_{j+1})$ , and we may repeat this process for each  $j$ . Thus we need only to show that this map is  $C^2$  at the fixed points.

Consider from Lemma 1 that as  $x$  approaches  $\bar{x}_{j+1}$  the map  $\tilde{u}$  mod 1 becomes uniformly close in the  $C^2$  sense to the map  $\xi$  defined by (2). Thus if  $x \in [a_i, a_{i+1})$ , then  $\phi$  is  $C^2$  close to the affine map

$$y = \frac{(b_{i+1} + b_i)x + a_{i+1}b_i - a_i b_{i+1}}{a_{i+1} - a_i}.$$

Let  $\bar{x}_{j+1}$  have local coordinate 0. The derivative of this map is

$$\frac{b_{i+1} - b_i}{a_{i+1} - a_i} = \frac{\beta b_i^2 + o(b_i^2)}{\alpha a_i^2 + o(a_i^2)},$$

where  $f(x) = x + \alpha x^2 + o(x^2)$  and  $g(x) = x + \beta x^2 + o(x^2)$ , and if this ratio approaches a nonzero limit as  $i \rightarrow \infty$  then  $\lim_{x \rightarrow 0^-} \phi'(x)$  exists. This is the case if we can show that  $a_i/b_i$  approaches a nonzero limit.

**Lemma 2.** *Choose  $a_0$  and  $b_0$  arbitrarily; then*

$$\lim_{i \rightarrow +\infty} \frac{a_i}{b_i} = \frac{\beta}{\alpha}.$$

Lemma 2 may be proven by bounding the orbits by orbits of appropriate differential equations. We omit the proof for the sake of brevity. Since the choices of  $a_0$  and  $b_0$  are arbitrary,  $\lim_{x \rightarrow 0^-} \phi'(x) = \alpha/\beta$ , and similarly,  $\lim_{x \rightarrow 0^+} \phi'(x) = \alpha/\beta$ . This implies that  $\phi$  is in fact differentiable at 0 and  $\phi'(0) = \alpha/\beta$ . Also, since  $\phi$  is  $C^2$  close to an affine map on each fundamental interval, we have that the limit of the second derivative from either side is zero. Therefore, the second derivative exists and is zero at 0, and so,  $\phi$  is a  $C^2$  at the fixed points.  $\square$

Theorem 1 and the fact that the forward and backward coordinates are  $C^2$  imply the following rigidity result.

**Corollary 2.** *Suppose  $f, g \in SN_n^2(X)$  are  $C^1$  conjugated; then they are  $C^2$  conjugated.*

Lemma 1 was introduced in [ALY] in the proof of the existence and smoothness of the conventional multipliers of homoclinic orbits, which we now describe in the context of  $f \in SN_1^2(\mathbf{S}^1)$ . Let 0 denote the fixed point of  $f$  and let  $[x_0]$  be the orbit through some point  $x_0$ . For each  $\ell > 0$ , define  $i_\ell^-$  to be the last integer such that  $x_\ell^- = x_{i_\ell^-} \in [0, \ell)$ , and  $i_\ell^+$  to be the first integer such that  $x_\ell^+ = x_{i_\ell^+} \in (-\ell, 0]$ . Consider the limit

$$b_f([x_0]) = \lim_{\ell \rightarrow 0} \frac{dx_\ell^+}{dx_\ell^-}.$$

**Theorem 2** ([ALY]). *The limit  $b_f([x_0])$  exists, is independent of coordinates and is continuously differentiable with respect to  $x_0$ .*

The quantity  $b_f([x_0])$  is called the **conventional multiplier** of the homoclinic orbit  $[x_0]$ , and it follows easily from the proof of Th. 2 in [ALY] that

$$b_f([x_0]) = DB_f([x_0]).$$

We note that this gives us an asymptotic formula for  $DB_f([x_0])$ , namely:

$$(5) \quad DB_f([x_0]) = \lim_{\ell \rightarrow 0} \prod_{i=i_\ell^-}^{i_\ell^+} Df(x_i).$$

Denote by  $D^2(\mathbf{S}^1)$  the set of  $C^2$  diffeomorphisms and by  $D_0^2(\mathbf{S}^1)$  the subset of  $D^2(\mathbf{S}^1)$  such that 0 is fixed. The set of standard maps is complete in the following sense.

**Lemma 3.** *For any  $\psi \in D_0^2$ , there is an  $f$  in  $SN_1^2(\mathbf{S}^1)$  such that  $\mathcal{B}_f \sim \psi$ .*

*Proof.* Let  $f \in SN_1^2(\mathbf{S}^1)$  be such that 0 is the fixed point and  $1/2$  is a degenerate critical point for  $\chi(x) \equiv f(x) - x$ , i.e.  $f'(1/2) = 1$  and  $f''(1/2) = 0$ . Choose  $a_0 = 1/2$  to serve as our reference point. First we will construct  $f_1$ , a modification of  $f$  for which  $\mathcal{B}_{f_1}$  has 1st and 2nd derivatives equal that of  $\psi$  at 0. By formula (5) this may be accomplished by modifying  $f$  in a small neighborhood of  $f(1/2)$  so  $f_1$  remains unaffected at other points of the orbit  $[1/2]$ . Specifically, we may choose the modification  $f_1$  to satisfy  $f_1'(f(1/2)) = (f'(f(1/2))/b_f([1/2]))\psi'(0)$ . We may make a similar choice to effect the desired change in the second derivative.

Next we will make a second modification  $f_2$  of  $f_1$  by inserting a new section of graph at the point  $1/2$ , namely,

$$f_2(x) = \begin{cases} f_1(x), & 0 \leq x < 1/2, \\ g(x), & 1/2 \leq x < f(1/2), \\ f_1(x - \chi(1/2)) + \chi(1/2), & f(1/2) \leq y \leq 1 + \chi(1/2), \end{cases}$$

where we choose  $g$  below. If we denote by  $\pi_{1,i} : \Gamma \rightarrow I_i$  the projection from the orbits of  $f_1$  onto each fundamental interval and by  $\pi_{2,i}$  the similar projection for  $f_2$ , then

$$(6) \quad \pi_{1,0} \circ (u_1^-)^{-1} \equiv \pi_{2,0} \circ (u_2^-)^{-1} \quad \text{and} \quad \pi_{1,1} \circ (u_1^+)^{-1} \equiv \pi_{2,1} \circ (u_2^+)^{-1} - \xi(1/2).$$

Now consider that  $\mathcal{B}_{f_2} = u^+ \circ \pi_{2,1}^{-1} \circ g \circ \pi_{2,0} \circ (u^-)^{-1}$ . Thus if we choose

$$g = \pi_{2,1} \circ (u^+)^{-1} \circ \psi \circ u^- \circ \pi_{2,0}^{-1}$$

then  $\mathcal{B}_{f_2} = \psi$ . Further, by (5) and (6) it is seen that  $g$  has unit derivatives and zero 2nd derivatives at  $1/2$  and  $f(1/2)$ , so that  $f_2$  is smooth at those points.  $\square$

Now consider diffeomorphisms of  $\mathbf{S}^1$  which have quadratic saddle-node periodic orbits  $\{\gamma_i\}_{i=1}^n$ . If  $p/q$  is the rotation number of  $f$  (see for example [KH] or [MS] for definition) then each  $\gamma_i$  consists of points  $\{x_{ij}\}_{j=1}^q$ . We may choose the index so that  $(x_{ij}, x_{i,j+1})$  contains no other periodic points and every point in  $(x_{ij}, x_{i,j+1})$  is heteroclinic from  $\gamma_j$  to  $\gamma_{j+1}$ . By quadratic saddle-node orbit we mean that  $x_{ij}$  is a quadratic saddle-node point of the diffeomorphism  $f^q$ . We can define forward and backward coordinates,  $u_{ij}^+$  and  $u_{ij}^-$ , for points in  $(x_{ij}, x_{i,j+1})$  under  $f^q$  as above and using this we may define a modulus for the interval by  $B_{ij} \equiv u_{ij}^+ \circ (u_{ij}^-)^{-1}$ .

**Lemma 4.** Fix  $j$ , and on each interval  $(x_{ij}, x_{i,j+1})$ ,  $1 \leq i \leq q$ , choose reference points  $a_{ij}$  such that  $f(a_{ij}) = a_{i+1,j}$ . Then

$$B_{ij} \equiv B_{kj} \text{ for all } 1 \leq i, k \leq q.$$

*Proof.* Let  $x_{ij}$  and  $x_{kj}$  be two periodic points of the orbit  $\gamma_j$ , and let  $l$  be the minimum integer so that  $f^l(x_{ij}) = x_{kj}$ . Now consider the forward coordinate of a point  $x \in (x_{i,j-1}, x_{ij})$ .

$$\begin{aligned} u_{ij}^+(x) &= \lim_{n \rightarrow \infty} \frac{f^{nq}(x) - f^{nq}(a_0)}{f^{(n+1)q}(a_0) - f^{nq}(a_0)} \\ &= \lim_{n \rightarrow \infty} \frac{f^{nq-l}(f^l(x)) - f^{nq-l}(f^l(a_0))}{f^{(n+1)q-l}(f^l(a_0)) - f^{nq-l}(f^l(a_0))}. \end{aligned}$$

Since  $f^{-l}$  is a diffeomorphism we obtain  $u_{ij}^+(x) = u_{kj}^+(f^l(x))$  and similarly  $u_{ij}^-(x) = u_{kj}^-(f^l(x))$ . Thus

$$\begin{aligned} B_{ij} &= u_{ij}^+ \circ (u_{ij}^-)^{-1} \\ &= u_{kj}^+ \circ f^l \circ (u_{kj}^- \circ f^l)^{-1} \\ &= B_{kj}. \end{aligned}$$

□

Now define  $SN_n^{2,p/q}$  to be the set of  $C^2$  diffeomorphisms of  $\mathbf{S}^1$  with rotation number  $p/q$  and  $n$  periodic orbits of quadratic type. Then a map  $f \in SN_n^{2,p/q}$  has moduli  $\mathcal{B}_f = \{B_1, \dots, B_n\}$  unique up to cyclic permutations and choices of base points. We again say that two such moduli are equivalent if the transition maps satisfy (3), and the following result generalizes Theorem 1.

**Corollary 3.**  *$f, g$  in  $SN_n^{2,p/q}$  are  $C^2$  conjugate if and only if  $\mathcal{B}_f \sim \mathcal{B}_g$ .*

Generalizations of Cor. 1 and Lem. 2 to the case of periodic points also hold.

### 3. TRANSITION MAPS OF HETEROCLINIC ORBITS OF HYPERBOLIC POINTS

If one considers the limit (2) for diffeomorphisms with hyperbolic fixed points one finds that the additive dependence on the based point of Lemma 1 fail to hold. Thus we introduce a modification of (2) which is suitable for hyperbolic fixed points.

Suppose that  $\bar{x}^+$  and  $\bar{x}^-$  are adjacent fixed points of  $f$ . Again, choose  $a_0 \in (\bar{x}^-, \bar{x}^+)$  and let  $a_{i+1} = f(a_i)$  and  $I_i = [a_i, a_{i+1})$ . Denote by  $\Gamma_j$  the set of orbits of  $f$  in  $(\bar{x}^-, \bar{x}^+)$ . Then given  $\gamma \in \Gamma_j$  let  $x_i = \gamma \cap I_i$  and consider the limits

$$(7) \quad u^\pm(x_0, a_0) = \lim_{i \rightarrow \pm\infty} \xi_i^\pm = \lim_{i \rightarrow \pm\infty} \frac{\log \frac{x_i - \bar{x}^\pm}{a_i - \bar{x}^\pm}}{\log \frac{a_{i+1} - \bar{x}^\pm}{a_i - \bar{x}^\pm}}.$$

**Lemma 5.** *Suppose that  $f \in C^k$ ,  $k \geq 2$ , and that  $\bar{x}_j$  and  $\bar{x}_{j+1}$  are hyperbolic fixed points. Then all the conclusions of Lemma 1 hold for (7) and the sequence  $\{\xi_i^\pm(\cdot)\}$  converges in the  $C^k$  topology on any interval  $I_i$ .*

*Proof.* If 0 is a hyperbolic fixed point of  $f$  and  $f$  is  $C^k$ ,  $k \geq 2$ , then  $f$  restricted to a neighborhood  $U$  of 0 is  $C^k$  conjugated to the linearization of  $f$  at 0. That is, there exist coordinates  $y$  on the neighborhood such that  $f$  is conjugated to the map

$$\bar{y} = f'(0)y.$$

In these coordinates the limit of  $\xi_i$  as  $i \rightarrow \infty$  exists trivially since  $\xi_i = \xi_0$ . Thus we in fact have existence and  $C^k$  smoothness of the limit for hyperbolic fixed points.

Next, suppose that a different coordinate chart in a neighborhood of 0 is given by coordinate  $\bar{x}$ , and  $\phi$  is the differentiable coordinate transformation. Then, using Taylor's Theorem at 0, we have for the ratio with respect to the new coordinates

$$(8) \quad \bar{\xi}_i = \frac{\log(x_i/a_i) + o(a_i)}{\log(a_{i+1}/a_i) + o(a_i)}.$$

And so, the limits are the same.

Next consider  $\bar{\xi}_i$  defined for a different choice of base point  $\bar{a}_0$ . Let us assume that  $x_0, a_0$ , and  $\bar{a}_0$  are close enough to the fixed point that we may use coordinates in which the map is linear. Then, since  $a_{i+1} = f'(0)a_i$ , and  $\bar{a}_{i+1} = f'(0)\bar{a}_i$

$$\begin{aligned} \bar{\xi}_i - \xi_i &= \frac{\log(x_i/\bar{a}_i)}{\log(\bar{a}_{i+1}/\bar{a}_i)} - \frac{\log(x_i/a_i)}{\log(a_{i+1}/a_i)} \\ &= \frac{\log \bar{a}_i - \log a_i}{\log f'(0)} \\ &= \frac{\log \bar{a}_0 - \log a_0}{\log f'(0)} \\ &= u^+([\bar{a}_0]). \end{aligned}$$

Since each pair of terms differs by a constant, the limits differ by the same constant, which is independent of  $x_0$ . □

It can be shown that these results hold also if  $\bar{x}^-$  or  $\bar{x}^+$  is a saddle-node. Again the limits  $u^\pm$  associate with each orbit points on  $[0, 1)$  and we may consider  $u^\pm$  as maps from  $\Gamma_j$  to the unit circle  $S^1$ . Denote by  $D^k(\lambda_1, \dots, \lambda_n; X)$  the set of  $C^k$  diffeomorphisms of  $X = \mathbf{R}$  or  $\mathbf{S}^1$ , with  $n$  hyperbolic fixed points  $\{\bar{x}_1, \dots, \bar{x}_n\}$  (in an adjacent ordering) with derivatives  $\{\lambda_1, \dots, \lambda_n\}$  at the respective fixed points. If  $X = \mathbf{S}^1$  then we consider cyclic permutations of  $\{\lambda_1, \dots, \lambda_n\}$  to be equivalent, or if  $X = \mathbf{R}$  then we consider  $\{\lambda_1, \dots, \lambda_n\}$  to be equivalent to  $\{\lambda_n, \dots, \lambda_1\}$ . It is well known that any two maps in  $D^k(\lambda_1, \dots, \lambda_n; X)$  are topologically conjugated.

As in Section 2, suppose that  $\bar{x}_j$  and  $\bar{x}_{j+1}$  are adjacent fixed points of  $f$  and let  $u_j^+$  and  $u_j^-$  denote the forward and backward coordinates of the orbits between  $\bar{x}_j$  and  $\bar{x}_{j+1}$ . The map  $B_j \equiv u_j^+ \circ (u_j^-)^{-1} : \mathbf{S}^1 \rightarrow \mathbf{S}^1$  we again call the **transition map** of the orbits between  $\bar{x}_j$ . The transition maps of two functions which satisfy Definition 3 we again call equivalent. The following may be easily shown by repeating the first part of the proof of Theorem 1.

**Proposition 1.** *If two  $C^k$  diffeomorphisms,  $k \geq 2$ ,  $f$  and  $g$  in  $D^k(\lambda_1, \dots, \lambda_n; X)$  are  $C^k$  conjugate then their moduli are equivalent.*

Now we answer the question of whether the moduli are sufficient for conjugacy in  $D^k(\lambda_1, \dots, \lambda_n; X)$ . The answer is yes, but only in a limited sense. In particular, suppose that  $\bar{x}_j$  and  $\bar{x}_{j+1}$  are adjacent hyperbolic fixed points of  $f$ , and  $\bar{y}_j$  and  $\bar{y}_{j+1}$  are adjacent hyperbolic fixed points of  $g$ . We may then repeat the steps of Theorem 1 to show:

**Theorem 3.** *The map  $f$  is  $C^2$  conjugated to  $g$  on an open extension of  $(\bar{x}_j, \bar{x}_{j+1})$  if and only if  $f'(\bar{x}_j) = g'(\bar{x}_j)$  and  $f'(\bar{x}_{j+1}) = g'(\bar{x}_{j+1})$  and  $B_{f,j} \sim B_{g,j}$ .*

By  $C^2$  conjugation on an open extension of  $(\bar{x}_j, \bar{x}_{j+1})$  we mean that there exists an open interval  $J$  containing  $[\bar{x}_j, \bar{x}_{j+1}]$ , an open interval  $K$  containing  $[\bar{y}_j, \bar{y}_{j+1}]$ , and a  $C^2$  diffeomorphism  $\phi : f(J) \rightarrow K$  such that (1) holds.

Thus we may say that the modulus completely determines conjugacy of heteroclinic orbits. However, there is a problem when trying to extend this theorem to all of  $X$  if  $X$  is  $\mathbf{S}^1$  or if  $X$  is  $\mathbf{R}$  and  $f$  and  $g$  have more than two fixed points. The problem is that of gluing the conjugating maps together smoothly at the fixed points.

For hyperbolic points, let  $\phi$  again be given by (4). Let  $i$  be sufficiently large that  $a_i$  and  $b_i$  are in neighborhoods of the fixed point in which  $f$  and  $g$  may be

linearized. Let each linearization have derivative 1 at the fixed points and let  $\bar{a}_i$  and  $\bar{b}_i$  be the linearized coordinates of  $a_i$  and  $b_i$ . We see that the one-sided limit of the derivative exists since

$$\bar{\phi}'(\bar{x}_j^\pm) = \lim_{i \rightarrow \pm\infty} \frac{\bar{a}_i}{\bar{b}_i} = \frac{\bar{a}_0}{\bar{b}_0} = s.$$

In the generic case for  $\mathcal{B}_f$ , each orbit of  $f$  must be mapped by the conjugation to a unique orbit of  $g$ . If we make a different choice of base points we may change  $\bar{a}_i$  and  $\bar{b}_i$  by at most a multiple of  $\lambda$ . Thus  $\bar{\phi}'(\bar{x}_j)$  can only assume a discrete countable set of values  $s\lambda^i$ , where  $i \in \mathbf{Z}$ .

Now suppose that  $\bar{x}_1, \bar{x}_2$  and  $\bar{x}_3$  are adjacent hyperbolic fixed points of  $f$  and  $\bar{y}_1, \bar{y}_2$ , and  $\bar{y}_3$  are adjacent hyperbolic fixed points of  $g$  such that  $f'(\bar{x}_j) = g'(\bar{y}_j)$  for  $j = 1, 2, 3$  and  $B_{f,j} = B_{g,j}$ ,  $j = 1, 2$ . Provided that  $B_{f,j}$  are not degenerate, then  $B_{f,j} = B_{g,j}$  will constrain the limit of  $\phi'$  at  $\bar{x}_2$  from the left to take on values  $\{s_1\lambda^i\}$  and from the right values  $\{s_2\lambda^i\}$ . Thus we can choose  $\phi$  to be differentiable at  $\bar{x}_2$  if and only if

$$s_1 = s_2\lambda^i,$$

for some  $i \in \mathbf{Z}$ . For diffeomorphisms of  $\mathbf{R}$  this condition must be satisfied for each adjacent triple of fixed points in order to have conjugacy. For diffeomorphisms of  $\mathbf{S}^1$ , the matching condition at the periodic points must always be satisfied.

**Proposition 2.** *There exist  $f$  and  $g$  in  $D^2(\lambda_1, \lambda_2; S^1)$  such that  $\mathcal{B}_f \sim \mathcal{B}_g$ , but  $f$  is not  $C^1$  conjugate to  $g$ .*

*Proof.* Let  $f : \mathbf{S}^1 \rightarrow \mathbf{S}^1$  be a  $C^k$  diffeomorphism with the following properties:

- (1) 0 and 1/2 are hyperbolic fixed points.
- (2) The transition maps of  $f$  on the intervals  $(0, 1/2)$  and  $(1/2, 1)$  have minimal period 1.
- (3)  $\chi(x) \equiv f(x) - x$  is positive on  $(0, 1/2)$  and negative on  $(1/2, 1)$ .
- (4) All the derivatives of  $\chi(x)$  up to order  $k$  vanish at 1/4.

The map  $g$  we define on the circle given by identifying the endpoints of the interval  $[0, 1 + \chi(1/4)]$  to be:

$$g(y) = \begin{cases} f(y), & 0 \leq y \leq 1/4, \\ y + \chi(1/4), & 1/4 \leq y \leq 1/4 + \chi(1/4), \\ f(y - \chi(1/4)), & 1/4 + \chi(1/4) \leq y \leq 1 + \chi(1/4). \end{cases}$$

It is clear that the functional moduli of  $f$  and  $g$  are identical. We may conjugate  $f$  and  $g$  on the interval  $[0, 1/4]$  by the identity function. The conjugacy has derivative 1 at the fixed point 0. This constrains the extension of this conjugacy to the interval  $[1/2, 1]$  to have right sided derivative 1 at  $x = 1/2$ . However, this also constrains the left sided derivative at 1/2 to have value  $f'(1/2)$ . Thus there is no conjugacy of this form. Moreover, there can be no conjugacy because any other prospective conjugacy can only have one-sided derivatives at 1/2 which differ by a factor of  $f'(1/2)$ . □

#### 4. DISCUSSION

In [NPT] there appeared a method for the study of saddle-node points in which the 1-d diffeomorphism is embedded in a 1-d flow. Provided that the original map is  $C^\infty$  then the flow is unique in the class  $C^\infty$ . This device has now become standard in the study of global bifurcations involving saddle-nodes. The drawbacks of this

method when used for  $C^k$  diffeomorphisms are that the vector field which defines the embedding flow is necessarily one degree less smooth than the diffeomorphism and even worse it is not known to be unique in  $C^{k-1}$ . For these reasons the embedding method is not suitable for the present setting.

Note that Theorem 1 only gives  $C^2$  conjugacy for  $C^2$  diffeomorphisms. It is natural to ask whether there is a  $C^k$  version of this theorem for  $2 < k \leq \infty$  or even in the class of analytic functions. The limiting factor for  $C^k$  is that the only known proof of Lemma 1 [ALY] does not lend itself to an induction step. However the results concerning embedding of  $C^\infty$  diffeomorphisms in flows imply that Lemma 1 and thus Theorem 1 are true in the class  $C^\infty$ . We also note here that the proof of Lemma 1 may be generalized to periodic points of higher degeneracy than 2.

In the case of hyperbolic periodic points the difficulty of extending smooth conjugacy beyond an interval bounded by the points seems to have been previously overlooked. Belitski's treatment sidesteps this issue completely by beginning with linearizations of the diffeomorphisms on whole neighborhoods of the fixed points rather than only on one side. The price paid for this is that the transition maps thus defined depend on two parameters rather than only one as in [KH] and here.

Finally the mixed case in which hyperbolic and nonhyperbolic periodic points coexist must be considered. As pointed out above, the limit (7) works for either type of fixed point. Thus this ratio may be used on an interval in which the endpoints are of different type and the transition map is defined as usual. Theorem 3 then holds on any such interval and for global conjugation one must then only check matching of the conjugacies at the hyperbolic periodic points.

#### ACKNOWLEDGMENT

The author wishes to thank V. Afraimovich for several helpful discussions.

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