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WEIGHTED WEAK-TYPE INEQUALITIES FOR THE MAXIMAL FUNCTION OF NONNEGATIVE INTEGRAL TRANSFORMS OVER APPROACH REGIONS

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Abstract. The relation between approach regions and singularities of nonnegative kernels $K_t(x,y)$ is studied, where $t \in (0,\infty)$, x, $y \in X$, and X is a homogeneous space. For $1 \leq p < q < \infty$, a sufficient condition on approach regions Ω_a $(a \in X)$ is given so that the maximal function

$$\sup_{(x,t)\in\Omega_a} \int_X K_t(x,y) f(y)\, d\sigma(y)$$

is weak-type (p,q) with respect to a pair of measures σ and ω. It is shown that this condition is also necessary for operators of potential type in the sense of Sawyer and Wheedon (Amer. J. Math. **114** (1992), 813–874).

1. Introduction

In [6], A. Nagel and E. Stein improved the classical theorem of Fatou on nontangential limits of harmonic functions defined on $\mathbb{R}^{n+1}_+$ to include limits within regions which allow sequential approach with any degree of tangency to the boundary, $\mathbb{R}^n$. Their proof is based on a remarkable result of a necessary and sufficient condition on the approach regions so that the associated maximal function of Hardy-Littlewood type is weak type $(1,1)$. The condition gives a clear picture of the relation between the size of cross-sectional measure of approach regions at height r and the "singularity", r^{-n}, of the maximal operator of Hardy-Littlewood type. Following this important work, a series of papers has been devoted to maximal functions of Hardy-Littlewood type for associated approach regions (see, e.g. [1], [9], [5], and [7]). In this paper, we show that a similar relation holds for general integral transforms with nonnegative kernels in the upper half-space of a homogeneous space. We give a sufficient condition, which turns out to be necessary for operators of potential type in the sense of [8], on approach regions so that weak-type (p,q) $(1 \leq p < q < \infty)$ inequalities (see (1.6) below) hold for the maximal function over the approach regions with respect to a pair of Borel measures σ and ω on X. Our condition and proof are motivated by a recent paper [3], where some general weak-type inequalities for integral transforms with nonnegative kernels are studied.

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A homogeneous space (X, d, μ) is a set X together with a quasi-metric d and a doubling measure μ. By the quasi-metric d we mean a mapping $d\colon X \times X \to [0, \infty)$ which satisfies the same conditions as a metric, except the triangle inequality is weakened to

$$d(x,y) \le \kappa(d(x,z) + d(z,y)) \qquad \text{for all } x,\ y,\ z \in X, \tag{1.1}$$

where $\kappa \ge 1$ is a constant which is independent of x, y, and z. Without loss of generality, we may assume that all balls $B(x,r) = \{y \in X\colon d(x,y) < r\}$ are open (see [4]). We also use the convention that $B(x,0) = \varnothing$. By the doubling measure μ we mean a nonnegative measure on the Borel subsets of X so that $|2B|_\mu \le C_\mu |B|_\mu$ for all balls $B \subset X$, where $|B|_\mu$ denotes the μ-measure of the ball B, and C_μ is a constant which depends only on μ.

For a homogeneous space X, we denote that $\widehat{X} = X \times (0, \infty)$ the upper half-space of $X \times \mathbb{R}$, and that $\widehat{B}(x,r) = B(x,r) \times (0,r)$. For a Borel measure σ on X, we study the operator T defined by

$$T(f\,d\sigma)(x,t) = \int_X K_t(x,y) f(y)\, d\sigma(y)\,, \qquad (x,t) \in \widehat{X}\,, \tag{1.2}$$

with the nonnegative kernel $K_t(x,y)$ which is lower semi-continuous.

Let $\Omega = \{\Omega_a\colon a \in X\}$ be a family of nonempty subsets in $\widehat{X}$. For each $a \in X$, set $\Omega_a(r) = \{x \in X\colon (x,r) \in \Omega_a\}$ (the cross-section of Ω_a at height r). Following [9], for $\alpha > 0$ and $(x,r) \in \widehat{X}$, we define

$$S_\alpha(x,r) = \{a \in X\colon \Omega_a(r) \cap B(x, \alpha r) \ne \varnothing\}\,. \tag{1.3}$$

We suppose that, for each $a \in X$, $a \in \Omega_a(r)$ for all $r > 0$, and $\Omega_a(r') \subset \Omega_a(r)$ if $r' \le r$. It is clear that, for each $x \in X$, $S_\alpha(x,r') \subset S_\alpha(x,r)$ if $r' \le r$, and $\lim_{r\to\infty} S_\alpha(x,r) = X$.

Theorem 1.1. *Let $1 \le p < q < \infty$ and $0 < \alpha < 2\kappa$. Assume that σ and ω are Borel measures on X, which satisfy the condition: There exists a positive constant C_1 such that*

$$\left| S_\alpha\left(x, \frac{4\kappa}{\alpha}(2r+t)\right)\right|_\omega^{1/q} \left(\int_{X\setminus B(x,r)} K_t(x,y)^{p'}\, d\sigma(y)\right)^{1/p'} \le C_1\,, \tag{1.4}$$

if $1 < p < \infty$, or

$$\left| S_\alpha\left(x, \frac{4\kappa}{\alpha}(2r+t)\right)\right|_\omega^{1/q} \operatorname{ess\,sup}_\sigma \big\{ K_t(x,y)\colon y \in X \setminus B(x,r)\big\} \le C_1\,, \tag{1.5}$$

if $p = 1$, for all $(x,t) \in \widehat{X}$ and $r \ge 0$, where $\operatorname{ess\,sup}_\sigma$ denotes the essential supremum with respect to the measure σ.

Then the weak-type inequality

$$\left| \left\{ a \in X\colon \sup_{(x,t)\in\Omega_a} T(f\,d\sigma)(x,t) > \lambda \right\}\right|_\omega^{1/q} \le \frac{C}{\lambda} \left(\int_X f(x)^p\, d\sigma(x)\right)^{1/p} \tag{1.6}$$

holds for all σ-measurable functions $f \ge 0$.

We shall say that an operator T defined by (1.2) is an operator of potential type if its kernel K satisfies the condition: For a fixed constant $0<\alpha<2\kappa$, there exists a positive constant C_0 such that

(1.7)
$$K_t(x,y)\le C_0 K_{t'}(x',y) \text{ whenever } (x',t')\in\widehat{B}\left(x,\frac{8\kappa^2}{\alpha}(r+t)\right) \text{ and } y\in X\setminus B(x,r)\,,$$

for all $(x,t)\in\widehat{X}$ and $r\ge 0$. (See also [8] for an essentially equivalent definition.)

Theorem 1.2. *Let $1\le p<q<\infty$ and $0<\alpha<2\kappa$, and let σ and ω be locally finite Borel measures on X. Assume that the kernel K satisfies condition* (1.7); *then the weak-type inequality* (1.6) *holds for all σ-measurable functions $f\ge 0$ if and only if there exists a positive constant C_1 such that*

$$\left|S_\alpha\left(x,\frac{4\kappa}{\alpha}\,t\right)\right|_\omega^{1/q}\left(\int_X K_t(x,y)^{p'}\,d\sigma(y)\right)^{1/p'}\le C_1\,, \tag{1.8}$$

if $1<p<\infty$, or

$$\left|S_\alpha\left(x,\frac{4\kappa}{\alpha}\,t\right)\right|_\omega^{1/q}\ \text{ess sup}_\sigma\big\{K_t(x,y)\colon y\in X\big\}\le C_1\,, \tag{1.9}$$

if $p=1$, for all $(x,t)\in\widehat{X}$.

2. Proof of Theorem 1.1

We shall assume that $1<p<\infty$. For $p=1$, only some mild modifications of the following proof are needed.

For any fixed function $f\ge 0$ and number $\lambda>0$, we set

$$E_\lambda=\left\{a\in X\colon \sup_{(x,t)\in\Omega_a} T(f\,d\sigma)(x,t)>\lambda\right\}. \tag{2.1}$$

Let B_R in X be a ball centered at some fixed $x_0\in X$ with radius $R>0$. We consider the sets

$$E^*_\lambda(R)=\left\{x\in B_R\colon \sup_{0<t\le R} T(f\,d\sigma)(x,t)>\lambda\right\}, \tag{2.2}$$

and

$$\begin{aligned} E_\lambda(R)=\big\{a\in X : T(f\,d\sigma)(x,t)>\lambda\ \text{ for some } (x,t)\in\Omega_a\,,\\ \text{with } x\in B_R \text{ and } 0<t\le d(x)\big\}\,, \end{aligned} \tag{2.3}$$

where

$$d(x)=\sup\big\{t\in(0,R]\colon T(f\,d\sigma)(x,t)>\lambda\big\}\,. \tag{2.4}$$

Obviously, we have $E^*_\lambda(R)\subset E_\lambda(R)\subset E_\lambda$, and $\lim_{R\to\infty}E_\lambda(R)=E_\lambda$.

We first claim that, for each $x\in E^*_\lambda(R)$ there exists $r(x)\ge(\alpha/2\kappa)d(x)$ (we recall that $\alpha<2\kappa$) such that

$$\left|S_\alpha\left(x,\frac{4\kappa}{\alpha}r(x)\right)\right|_\omega\le\frac{C}{\lambda^q}\,\|f\|^{q-p}_{L^p(d\sigma)}\int_{B(x,r(x))}f(y)^p\,d\sigma(y)\,, \tag{2.5}$$

where the constant C is independent of x and R.

Assuming the last claim for a moment, we finish the proof of the theorem. We may assume that $\sup\{r(x) \colon x \in E^*_\lambda(R)\} < \infty$; otherwise (1.6) would follow trivially from

$$|X|_\omega = \sup_{x\in E^*_\lambda(R)} \left| S_\alpha\left(x, \frac{4\kappa}{\alpha} r(x)\right)\right|_\omega \le \frac{C}{\lambda^q} \|f\|^{q-p}_{L^p(d\sigma)} \int_X f(x)^p \, d\sigma(x) \,.$$

Applying the covering lemma for homogeneous spaces (see [2]) with the family of balls $\{B(x, r(x)) \colon x \in E^*_\lambda(R)\}$, we obtain a countable, pairwise disjoint subfamily $\{B(x_k, r_k)\}_{k=0}^\infty$, where $r_k = r(x_k)$, so that $E^*_\lambda(R) \subset \bigcup_k B(x_k, 4\kappa r_k)$, and $r(x) \le 2r_k$ for all $x \in B(x_k, 4\kappa r_k)$.

We next claim that

$$E_\lambda(R) \subset \bigcup_k S_\alpha\left(x_k, \frac{4\kappa}{\alpha} r_k\right) . \tag{2.6}$$

Indeed, if $a \in E_\lambda(R)$ then there is $(x, t) \in \Omega_a$ so that $x \in E^*_\lambda(R)$ and $t \le d(x)$. Thus, $x \in B(x_k, 4\kappa r_k)$ and $t \le d(x) \le (2\kappa/\alpha) r(x) \le (4\kappa/\alpha) r_k$ for some k. Therefore, $(x, (4\kappa/\alpha) r_k) \in \Omega_a$ by our assumption on Ω. This implies that

$$x \in \Omega_a\left(\frac{4\kappa}{\alpha} r_k\right) \cap B\left(x_k, \alpha\frac{4\kappa}{\alpha} r_k\right),$$

and therefore, $a \in S_\alpha(x_k, (4\kappa/\alpha) r_k)$.

By virtue of (2.6) and (2.5), we have

$$\begin{aligned}
|E_\lambda(R)|_\omega &\le \sum_k \left| S_\alpha\left(x_k, \frac{4\kappa}{\alpha} r_k\right)\right|_\omega \\
&\le \frac{C}{\lambda^q} \|f\|^{q-p}_{L^p(d\sigma)} \sum_k \int_{B(x_k, r_k)} f(x)^p \, d\sigma(x) \\
&\le \frac{C}{\lambda^q} \|f\|^{q-p}_{L^p(d\sigma)} \int_X f(x)^p \, d\sigma(x) \\
&\le \frac{C}{\lambda^q} \left(\int_X f(x)^p \, d\sigma(x)\right)^{q/p} ,
\end{aligned}$$

where we have used the fact that $\{B(x_k, r_k)\}$ is pairwise disjoint. By letting $R \to \infty$, we obtain (1.6).

We now prove (2.5). If $x \in E^*_\lambda(R)$, then there exists t with $(\alpha/2\kappa) d(x) \le t \le d(x)$ so that $T(f\, d\sigma)(x, t) > \lambda$. To simplify the notation, we shall denote $S(r) = S_\alpha(x, (4\kappa/\alpha) r)$. Without loss of generality, we can assume that there is $r > 0$ so that

$$\frac{C_1}{|S(2r+t)|^{1/q}_\omega} \left(\int_X \left(\frac{2f(y)}{\lambda}\right)^p d\sigma(y)\right)^{1/p} < 1 \,. \tag{2.7}$$

Otherwise, (1.6) would follow trivially from

$$|X|^{1/q}_\omega = \lim_{r\to\infty} |S(2r+t)|^{1/q}_\omega \le \frac{2C_1}{\lambda} \left(\int_X f(y)^p d\sigma(y)\right)^{1/p} .$$

Let d_0 be the greatest lower bound for the r satisfying (2.7); then, by using Hölder's inequality and condition (1.4), we have

$$\begin{aligned}\int_{X\setminus B(x,r)} K_t(x,y)f(y)\,d\sigma(y) &= \frac{\lambda}{2}\int_{X\setminus B(x,r)} K_t(x,y)\frac{2f(y)}{\lambda}\,d\sigma(y)\\ \leq \frac{\lambda}{2}\left(\int_{X\setminus B(x,r)} K_t(x,y)^{p'}\,d\sigma(y)\right)^{1/p'} &\left(\int_{X\setminus B(x,r)}\left(\frac{2f(y)}{\lambda}\right)^p d\sigma(y)\right)^{1/p}\\ \leq \frac{C_1\lambda}{2|S(2r+t)|_\omega^{1/q}}\left(\int_X\left(\frac{2f(y)}{\lambda}\right)^p d\sigma(y)\right)^{1/p} &\leq \frac{\lambda}{2}\,,\end{aligned}$$

for all $r > d_0$. Therefore, we obtain either

$$\int_{B(x,r)} K_t(x,y)f(y)\,d\sigma(y) > \frac{\lambda}{2}\,, \qquad \text{if } \ r > d_0 > 0\,, \tag{2.8}$$

or

$$K_t(x,x)f(x)|\{x\}|_\sigma > \frac{\lambda}{2}\,, \qquad \text{if } \ d_0 = 0\,. \tag{2.9}$$

If $d_0 = 0$ then

$$\begin{aligned}\frac{\lambda}{2} < K_t(x,x)f(x)|\{x\}|_\sigma &\leq \int_{B(x,t)} K_t(x,y)f(y)\,d\sigma(y)\\ &\leq \left(\int_X K_t(x,y)^{p'}\,d\sigma(y)\right)^{1/p'}\left(\int_{B(x,t)} f(y)^p\,d\sigma(y)\right)^{1/p}\\ &\leq \frac{C_1}{|S(t)|_\omega^{1/q}}\left(\int_{B(x,t)} f(y)^p\,d\sigma(y)\right)^{1/p},\end{aligned}$$

and so that

$$\begin{aligned}|S(t)|_\omega &\leq \left(\frac{2C_1}{\lambda}\right)^q\left(\int_{B(x,t)} f(y)^p\,d\sigma(y)\right)^{q/p}\\ &\leq \frac{C}{\lambda^q}\,\|f\|_{L^p(d\sigma)}^{q-p}\int_{B(x,t)} f(y)^p\,d\sigma(y)\,.\end{aligned}$$

If $d_0 > 0$ then there is $r_0 > 0$ such that

$$\frac{C_1}{|S(2r_0+t)|_\omega^{1/q}}\left(\int_X\left(\frac{2f(y)}{\lambda}\right)^p d\sigma(y)\right)^{1/p} < 1\,, \tag{2.10}$$

and

$$\frac{C_1}{|S(r_0+t)|_\omega^{1/q}}\left(\int_X\left(\frac{2f(y)}{\lambda}\right)^p d\sigma(y)\right)^{1/p} \geq 1\,. \tag{2.11}$$

We now choose a (finite or infinite) decreasing sequence $\{r_j\}_{j=1}^n$ of positive numbers such that

$$|S(r_j+t)|_\omega \leq 2^{-j}|S(r_0+t)|_\omega \leq |S(2r_j+t)|_\omega \tag{2.12}$$

for $j = 1, 2, \ldots, n$. Due to $p < q$, we have

$$\sum_{j=1}^{n} \left(\frac{|S(r_j + t)|_\omega}{|S(r_0 + t)|_\omega} \right)^{1/p-1/q} \leq \sum_{j=1}^{\infty} \left(\frac{1}{2^j} \right)^{1/p-1/q} = \beta < \infty .$$

Thus, we obtain from (2.8) that

$$\frac{\lambda}{2\beta} \sum_{j=1}^{n} \left(\frac{|S(r_j + t)|_\omega}{|S(r_0 + t)|_\omega} \right)^{1/p-1/q} \leq \sum_{j=1}^{n} \int_{B_j \setminus B_{j+1}} K_t(x, y) f(y) \, d\sigma(y) ,$$

where $B_j = B(x, r_j)$, and $B_{n+1} = \{x\}$ if n is finite. Hence, there exists j_0 such that

$$\begin{aligned} \frac{\lambda}{2\beta} \left(\frac{|S(r_{j_0} + t)|_\omega}{|S(r_0 + t)|_\omega} \right)^{1/p-1/q} &< \int_{B_{j_0} \setminus B_{j_0+1}} K_t(x, y) f(y) \, d\sigma(y) \\ \leq \left(\int_{B_{j_0} \setminus B_{j_0+1}} K_t(x, y)^{p'} \, d\sigma(y) \right)^{1/p'} & \left(\int_{B_{j_0} \setminus B_{j_0+1}} f(y)^p \, d\sigma(y) \right)^{1/p} \\ \leq \frac{C_1}{|S(2r_{j_0+1} + t)|_\omega^{1/q}} & \left(\int_{B_{j_0}} f(y)^p \, d\sigma(y) \right)^{1/p} \\ \leq \frac{2^{1/q} C_1}{|S(r_{j_0} + t)|_\omega^{1/q}} & \left(\int_{B_{j_0}} f(y)^p \, d\sigma(y) \right)^{1/p} , \end{aligned}$$

the last step follows from (2.12). Consequently, by using (2.11), we get

$$\begin{aligned} |S(r_{j_0} + t)|_\omega &\leq \frac{C}{\lambda^p} |S(r_0 + t)|_\omega^{1-p/q} \int_{B(x, r_{j_0})} f(y)^p \, d\sigma(y) \\ &\leq \frac{C}{\lambda^q} \|f\|_{L^p(d\sigma)}^{q-p} \int_{B(x, r_{j_0})} f(y)^p \, d\sigma(y) \\ &\leq \frac{C}{\lambda^q} \|f\|_{L^p(d\sigma)}^{q-p} \int_{B(x, r_{j_0} + t)} f(y)^p \, d\sigma(y) . \end{aligned}$$

This completes the proof of (2.5), if we choose $r(x) = r_{j_0} + t$ and note $t \geq (\alpha/2\kappa) d(x)$. Therefore Theorem 1.1 is proved.

3. Proof of Theorem 1.2

We first show that conditions (1.8) and (1.9) imply conditions (1.4) and (1.5), respectively, if the kernel K satisfies condition (1.7). Then the sufficient part of the theorem follows from Theorem 1.1. To see this, we fix $x \in X$, $t > 0$ and $r \geq 0$. Then, $(x, 2r + t) \in \widehat{B}(x, (8\kappa^2/\alpha)(r + t))$, since $0 < \alpha < 2\kappa$, and so that

$$K_t(x, y) \leq C_0 K_{2r+t}(x, y)$$

for $y \in X \setminus B(x,r)$, by (1.7). Therefore

$$\begin{aligned}
&\left|S_\alpha\left(x, \frac{4\kappa}{\alpha}(2r+t)\right)\right|_\omega^{1/q} \left(\int_{X\setminus B(x,r)} K_t(x,y)^{p'}\, d\sigma(y)\right)^{1/p'} \\
&\le \left|S_\alpha\left(x, \frac{4\kappa}{\alpha}(2r+t)\right)\right|_\omega^{1/q} \left(\int_{X\setminus B(x,r)} K_{2r+t}(x,y)^{p'}\, d\sigma(y)\right)^{1/p'} \\
&\le \left|S_\alpha\left(x, \frac{4\kappa}{\alpha}\, t\right)\right|_\omega^{1/q} \left(\int_X K_t(x,y)^{p'}\, d\sigma(y)\right)^{1/p'} \le C_1\, .
\end{aligned}$$

The proof for the case $p = 1$ is similar.

We now prove the necessity part of the theorem. We consider the case of $1 < p < \infty$ first. Let $(x_0, t_0) \in \widehat{X}$ be temporarily fixed, and for each positive integer N, define

$$K_{t_0}^N(x_0, y) = \begin{cases} K_{t_0}(x_0, y) & \text{if } K(x_0, y) < N, \\ N & \text{if } K_{t_0}(x_0, y) \ge N. \end{cases}$$

Let $R > 0$ be arbitrary fixed constant and $0 < \beta < 1/C_0$, we now take

$$f(y) = K_{t_0}^N(x_0, y)^{p'/p} \chi_{B(x_0,R)}(y)$$

and $\lambda = \beta \int_{B(x_0,R)} K_{t_0}^N(x_0,y)^{p'}\, d\sigma(y)$. We note that $\lambda < \infty$ since σ is locally finite. If $a \in S_\alpha(x_0, (4\kappa/\alpha)t_0)$, then there is $x \in \Omega_a((4\kappa/\alpha)t_0) \cap B(x_0, 4\kappa t_0)$, and hence $(x, (4\kappa/\alpha)t_0) \in \Omega_a \cap \widehat{B}(x_0, (8\kappa^2/\alpha)t_0)$, since $0 < \alpha < 2\kappa$. Apply condition (1.7) with $r = 0$, we get $K_{t_0}(x_0, y) \le C_0 K_{(4\kappa/\alpha)t_0}(x, y)$ for all $y \in X$, and therefore, $K_{t_0}^N(x_0, y) \le C_0 K_{(4\kappa/\alpha)t_0}(x, y)$ for all $y \in X$. Thus

$$T(f\, d\sigma)\left(x, \frac{4\kappa}{\alpha}\, t_0\right) \ge \frac{1}{C_0} \int_{B(x_0,R)} K_{t_0}^N(x_0,y) K_{t_0}^N(x_0,y)^{p'/p}\, d\sigma(y) > \lambda\, ,$$

which implies that

$$S_\alpha\left(x_0, \frac{4\kappa}{\alpha}\, t_0\right) \subset \left\{ a \in X \colon \sup_{(x,t)\in\Omega_a} T(f\, d\sigma)(x,t) > \lambda \right\}\, .$$

Therefore, by using the weak-type inequality (1.6), we obtain

$$\begin{aligned}
&\beta \left|S_\alpha\left(x_0, \frac{4\kappa}{\alpha}\, t_0\right)\right|_\omega^{1/q} \int_{B(x_0,R)} K_{t_0}^N(x_0,y)^{p'}\, d\sigma(y) \\
&\le \lambda \left|\left\{ a \in X \colon \sup_{(x,t)\in\Omega_a} T(f\, d\sigma)(x,t) > \lambda \right\}\right|_\omega^{1/q} \\
&\le C \left(\int_{B(x_0,R)} K_{t_0}^N(x_0,y)^{p'}\, d\sigma(y)\right)^{1/p}\, .
\end{aligned}$$

Thus, (1.8) follows by division, and then letting $R \to \infty$ and $N \to \infty$.

For the case $p = 1$, we fix $(x_0, t_0) \in \widehat{X}$ temporarily, and for each positive integer $N > 4\kappa t_0$ set

$$\Lambda_N = \min\left\{N\, ,\ \operatorname{ess\,sup}_\sigma\left\{K_{t_0}(x_0,y) \colon y \in X\right\}\right\}\, .$$

We could assume that $\Lambda_N > 0$, otherwise (1.9) would hold trivially. Let R be an arbitrary positive number, and choose a number $0 < \eta < 1$ so that the set

$$U_{R,N} = \{y \in B(x_0, N) \colon K_{t_0}(x_0, y) \geq \eta \Lambda_N\}$$

has nonzero and finite σ-measure, since σ is locally finite. Now, take $f = \chi_{U_{R,N}}$ and $\lambda = \beta \Lambda_N |U_{R,N}|_\sigma$ with $0 < \beta < \eta/C_0$. As before, if $a \in S_\alpha(x_0, (4\kappa/\alpha)t_0)$, then there is $x \in \Omega_a((4\kappa/\alpha)t_0) \cap B(x_0, 4\kappa t_0)$ and hence condition (1.7) implies that

$$T(f\, d\sigma)\left(x, \frac{4\kappa}{\alpha}\, t_0\right) \geq \frac{1}{C_0} \int_X K_{t_0}(x_0, y) \chi_{U_{R,N}}(y)\, d\sigma(y) \geq \frac{\eta}{C_0} \Lambda_N |U_{R,N}|_\sigma > \lambda\,,$$

since $N > 4\kappa t_0$. Therefore, by the weak-type inequality (1.6), we obtain

$$\begin{aligned} &\beta \left| S_\alpha\left(x_0, \frac{4\kappa}{\alpha}\, t_0\right)\right|_\omega^{1/q} \Lambda\, |U_{R,N}|_\sigma \\ &\leq \lambda \left| \left\{ a \in X \colon \sup_{(x,t) \in \Omega_a} T(f\, d\sigma)(x,t) > \lambda \right\} \right|_\omega^{1/q} \\ &\leq C \left(\int_X \chi_{U_{R,N}}(y)\, d\sigma(y) \right) = C |U_{R,N}|_\sigma\,. \end{aligned}$$

Again, (1.9) follows by division, and then letting $R \to \infty$ and $N \to \infty$. This completes the proof of Theorem 1.2.

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