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ON WEIGHTED INEQUALITIES FOR SINGULAR INTEGRALS

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Abstract. In this note we consider singular integrals associated to Calderón-Zygmund kernels. We prove that if the kernel is supported in $(0,\infty)$ then the one-sided A_p condition, A_p^-, is a sufficient condition for the singular integral to be bounded in $L^p(w)$, $1<p<\infty$, or from $L^1(wdx)$ into weak-$L^1(wdx)$ if $p=1$. This one-sided A_p condition becomes also necessary when we require the uniform boundedness of the singular integrals associated to the dilations of a kernel which is not identically zero in $(0,\infty)$. The two-sided version of this result is also obtained: Muckenhoupt's A_p condition is necessary for the uniform boundedness of the singular integrals associated to the dilations of a general Calderón-Zygmund kernel which is not the function zero either in $(-\infty,0)$ or in $(0,\infty)$.

Introduction

It is a classical result in the theory of weighted inequalities the fact that the A_p condition of B. Muckenhoupt on a weight w is equivalent to the $L^p(wdx)$ boundedness of the Hilbert transform. This result was proved in 1973 by Hunt, Muckenhoupt and Wheeden [HMW]. In 1974 Coifman and Fefferman [CF] gave a different proof which relies on a good–λ inequality, producing an integral estimate of the singular integral in terms of the Hardy–Littlewood maximal operator.

Since 1986 the work by E. Sawyer [S], Andersen and Sawyer [AS], Martín Reyes, Ortega Salvador and de la Torre [MOT], [MT] has shown that many positive operators of real analysis have one-sided versions for which the classes of weights are larger than Muckenhoupt's ones. Our purpose here is to study the corresponding problems for singular integrals.

The situation for one–sided singular integrals is different. The symmetry properties of the Hilbert kernel produce the necessary cancellation properties of a singular integral, so that, no one–sided truncation of $1/x$ is expected to produce a one-sided singular integral. Nevertheless, as we show in Lemma (1.5), the class of general singular integral Calderón–Zygmund kernels supported on a half line is nontrivial. We ask for the more general class of weights w for which such singular integral operators are bounded in $L^p(wdx)$. It turns out (Theorem (2.1)) that the one–sided A_p condition is a sufficient condition which becomes also necessary when we require

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the uniform boundedness of the singular integrals associated to the dilations of a kernel which is not the function zero in $(0,\infty)$ or in $(-\infty,0)$ (Theorem (2.6)). The two-sided version of this result gives (see also Theorem (2.6)) that Muckenhoupt's A_p condition is necessary for the uniform boundedness of the singular integrals associated to the dilations of a Calderón-Zygmund kernel which is not the function zero either in $(-\infty,0)$ or in $(0,\infty)$.

1. One-sided singular integrals

We shall say that a function k in $L^1_{loc}(\mathbb{R}-\{0\})$ is a Calderón–Zygmund kernel if the following properties are satisfied:

(1.1) there exists a finite constant B_1 such that

$$\left|\int_{\varepsilon<|x|<N} k(x)\,dx\right| \le B_1 \quad \text{for all } \varepsilon \text{ and all } N, \text{ with } 0<\varepsilon<N,$$

and furthermore $\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x|<1} k(x)\,dx$ exists,

(1.2) there exists a finite constant B_2 such that

$$|k(x)| \le \frac{B_2}{|x|}, \quad \text{for all } x \ne 0,$$

(1.3) there exists a finite constant B_3 such that

$$|k(x-y)-k(x)| \le B_3|y||x|^{-2} \quad \text{for all } x \text{ and } y \text{ with } |x|>2|y|>0.$$

These conditions are known to be sufficient for the L^p boundedness and the weak type $(1,1)$ of the maximal operator

$$T^*f(x) = \sup_{\varepsilon>0}|T_\varepsilon f(x)|,$$

with

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x-y)f(y)\,dy$$

(see e.g. [T]). In other words

$$|\{x\in\mathbb{R} : T^*f(x)>\lambda\}| \le \frac{C_1}{\lambda}\int_{\mathbb{R}}|f|$$

and

$$||T^*f||_p \le C_p||f||_p$$

for $1<p<\infty$, where $|E|$ stands for the Lebesgue measure of the set E and $\|f\|_p = (\int |f(x)|^p\,dx)^{1/p}$. The same inequalities hold for the singular integral

$$Tf(x) = \lim_{\varepsilon\to 0^+} T_\varepsilon f(x).$$

The next lemma shows that there exist nontrivial Calderón-Zygmund kernels with support contained in $(0,\infty)$.

(1.4) Lemma. *The function*

$$k(x) = \frac{1}{x}\cdot\frac{\sin(\log x)}{\log x}\cdot\chi_{(0,\infty)}(x), \tag{1.5}$$

is a Calderón-Zygmund kernel.

Proof. To prove (1.1), take $0 < \varepsilon < N < \infty$ and change variables in the integral $\int_{\varepsilon<|x|<N} k(x)\,dx$ to get

$$\int_{\log\varepsilon}^{\log N} \frac{\sin t}{t}\,dt,$$

which is uniformly bounded in ε and N and, for fixed N, converges to a finite limit as ε tends to zero. On the other hand, since the function $g(t) = t^{-1}\sin t$ is bounded by one, it is clear that k satisfies (1.2) with $B_2 = 1$. Let us prove (1.3). Notice that the function $g(t)$ has a bounded derivative so that

$$\frac{d}{dx}k(x) = \frac{d}{dx}\,\frac{1}{x}g(\log x)\} = \frac{1}{x^2}\{g'(\log x) - g(\log x)\},$$

from which (1.3) follows with a constant B_3. □

2. Weighted inequalities

The one-sided Hardy-Littlewood maximal functions are defined by

$$M^+f(x) = \sup_{h>0}\frac{1}{h}\int_x^{x+h}|f| \quad \text{and} \quad M^-f(x) = \sup_{h>0}\frac{1}{h}\int_{x-h}^{x}|f|.$$

It is a known result by E. Sawyer [S] that M^- applies $L^p(wdx)$ into $L^p(wdx)$, $1 < p < \infty$, if and only if the weight w satisfies

$$A_p^- : \left(\int_a^b w^{-\frac{1}{p-1}}\right)^{p-1}\left(\int_b^c w\right) \le C(c-a)^p$$

for all numbers $a < b < c$ with a finite constant C independent of a, b, and c.

Also M^- applies $L^1(wdx)$ into weak-$L^1(wdx)$ if and only if w satisfies

$$A_1^- : M^+w \le Cw \quad \text{a.e.}$$

Analogous results hold for the operator M^+ and the corresponding A_p^+ classes which are defined in the obvious way.

In the more recent paper [MPT] the one-sided analog of A_∞ weights are studied. A weight w is in A_∞^- if and only if there exist positive numbers C and δ such that for all numbers $a < b < c$ and all measurable sets $E \subset (a, b)$

$$\frac{|E|}{c-a} \le C\left(\frac{w(E)}{w(b,c)}\right)^\delta$$

(as usual, $w(E)$ stands for $\int_E w(x)\,dx$). This class of weights is proved to coincide with the union of the A_p^- classes with $p \ge 1$. In fact, we have the following result:

Theorem A [MPT, Theorem 1]**.** *Let w be a weight, i.e., a positive, locally integrable function in the real line. The following are equivalent:*

(a) *There exists p, $1 \le p < \infty$, such that $w \in A_p^-$.*
(b) *w is in A_∞^-.*
(c) *There exist positive numbers C and δ such that*

$$\frac{w(E)}{w(a,c)} \le C\left(\frac{|E|}{b-a}\right)^\delta$$

for all numbers $a < b < c$ and all measurable sets $E \subset (b, c)$.

We are now in position to state the main results of this note.

(2.1) Theorem. *Let k be a Calderón-Zygmund kernel with support in $\mathbb{R}^+ = (0,\infty)$. Then*

(2.2) *given a weight w in A_∞^- there exists a constant C_p depending only on B_1, B_2, B_3, p and the constants in the A_∞^- condition, such that*

$$\int_{\mathbb{R}} |T^* f(x)|^p w(x)\,dx \le C_p \int_{\mathbb{R}} |M^- f(x)|^p w(x) dx, \quad 1 < p < \infty,$$

and

$$\sup_{\lambda>0} \lambda^p w(\{x : T^* f(x) > \lambda\}) \le C_p \sup_{\lambda>0} \lambda^p w(\{x : M^- f(x) > \lambda\}), \quad 1 \le p < \infty,$$

for all $f \in L^p(wdx)$,

(2.3) *given a weight $w \in A_p^-$ with $1 < p < \infty$ there exists a constant C depending only on B_1, B_2, B_3, p and the constant in the condition A_p^-, such that*

$$\int_{\mathbb{R}} |T^* f(x)|^p w(x)\,dx \le C \int_{\mathbb{R}} |f(x)|^p w(x)\,dx,$$

for all $f \in L^p(wdx)$,

(2.4) *given a weight $w \in A_1^-$ there exists a constant C depending only on B_1, B_2, B_3 and the constant in the condition A_1^- such that*

$$w(\{x : T^* f(x) > \lambda\}) \le \frac{C}{\lambda} \int_{\mathbb{R}} |f(x)| w(x)\,dx$$

for all $f \in L^1(wdx)$ and all $\lambda > 0$.

(2.5) Remarks. Let us remark here that similar right-hand-sided results hold with kernels supported on the negative real numbers, changing M^- to M^+ and A_p^- to A_p^+.

Let k be a Calderón-Zygmund kernel and let us consider for each $\alpha > 0$ the dilation

$$k_\alpha(x) = \alpha k(\alpha x).$$

It is clear that k_α is also a Calderón-Zygmund kernel with the same constants B_1, B_2 and B_3 as k. Therefore, Theorem (2.1) gives that the maximal singular integrals T_α^* associated to the dilations k_α are uniformly bounded in $L^p(wdx)$ if $w \in A_p^-$ with $1 < p < \infty$ and from $L^1(wdx)$ into weak-$L^1(wdx)$ if $w \in A_1^-$. The next theorem is a kind of converse of this remark.

(2.6) Theorem. *Let k be a Calderón-Zygmund kernel. For each $\alpha > 0$, let T_α^* denote the maximal operator with kernel $k_\alpha(x) = \alpha k(\alpha x)$ and let $1 \le p < \infty$. Let w be a weight and assume that the operators T_α^* are uniformly bounded from $L^p(wdx)$ into weak-$L^p(wdx)$.*

(a) *If there exists $z > 0$ such that $k(z) \ne 0$ then $w \in A_p^-$.*
(b) *If there exists $s < 0$ such that $k(s) \ne 0$ then $w \in A_p^+$.*
(c) *If there exist $s < 0 < z$ such that $k(s) \ne 0 \ne k(z)$ then $w \in A_p$; i.e., w belongs to the A_p class of Muckenhoupt.*

Theorems (2.1) and (2.6) hold also for the singular integral

$$Tf(x) = \lim_{\varepsilon\to 0^+} T_\varepsilon f(x).$$

The proofs for T are similar to the corresponding for T^* or follow easily from the theorem for T^*.

Proof of Theorem (2.1). Theorem (2.1) is an easy consequence of Sawyer's results for M^- [S] and of the next lemma which is itself an extension, to the one-sided setting, of the good-λ inequality of Coifman and Fefferman [CF]. □

(2.7) Lemma. *Let k be a Calderón-Zygmund kernel with support in $\mathbb{R}^+$. Let w be a weight in A_∞^-. Then there exist positive constants C and γ_o such that for every $0<\gamma\le\gamma_o$ the inequality*

$$w(\{x\in\mathbb{R} : T^*f(x) > 2\lambda, M^-f(x) < \gamma\lambda\}) \le C\gamma^\delta w(\{x\in\mathbb{R} : T^*f(x) > \lambda\}), \tag{2.8}$$

$f\in L^1$ and every positive λ with δ the exponent in statement (c) *of Theorem* A.

Proof of (2.7). Since the set $\{x : T^*f(x) > \lambda\}$ is open and has finite measure for f in L^1, we have that it can be written as a disjoint countable union of open intervals. Let $J=(a,b)$ be such an interval. It is enough to prove that there exist C and γ_0 such that

$$w(\{x\in J : T^*f(x) > 2\lambda, M^-f(x) < \gamma\lambda\}) \le C\gamma^\delta w(J), \tag{2.9}$$

for every $0<\gamma\le\gamma_0$ and every $\lambda>0$ (throughout the proof, C will be a constant that may change from line to line.) Let us now take a sequence $\{x_i : i\in\mathbb{N}\}$ in $J=(a,b)$ in such a way that $x_0=b$ and $x_{i-1}-x_i = x_i - a$ for every $i\ge 1$. Since the weight $w\in A_\infty^-$ we have by Theorem A (statement (c) that, in order to prove (2.9), we only need to show that

$$|E_i| \le C\gamma(x_{i+1}-x_{i+2}), \tag{2.10}$$

where $E_i = \{x\in(x_{i+1},x_i) : T^*f(x) > 2\lambda, M^-f(x) < \gamma\lambda\}$. In fact, if (2.10) holds then by (c) in Theorem A we have that

$$w(E_i) \le C\gamma^\delta w(x_{i+2},x_i).$$

Summing up in i we obtain (2.9).

In order to prove (2.10) we fix $i\in\mathbb{N}$ and choose $\bar a < a$ in such a way that $x_i - a = a-\bar a$. Now decompose f as f_1+f_2 with $f_1 = f\chi_{(\bar a,\infty)}$. If $E_i=\emptyset$ there is nothing to prove. Let us write $\xi_0 = \sup\{\xi\in(x_{i+1},x_i) : M^-f(\xi)\le\gamma\lambda\}$ and notice that

$$\begin{aligned} E_i \subset &\{x\in(x_{i+1},\xi_0) : T^*f_1(x) > \lambda/2\} \\ &\cup \{x\in(x_{i+1},x_i) : T^*f_2(x) > \frac{3}{2}\lambda,\ M^-f(x)\le\gamma\lambda\}. \end{aligned} \tag{2.11}$$

Since for $x\in(x_{i+1},\xi_0)$, $T^*f_1(x) = T^*(f\chi_{(\bar a,\xi_0)})(x)$ because of the support property on k, applying the weak type $(1,1)$ of T^* we have the desired bound for the measure of the first term on the right of (2.11):

$$\begin{aligned} |\{(x_{i+1},\xi_0) : T^*f_1 > \lambda/2\}| &\le \frac{C}{\lambda}\int_{\bar a}^{\xi_0}|f| \\ &\le \frac{C}{\lambda}(\xi_0-\bar a)M^-f(\xi_0) \le C\gamma(x_{i+1}-x_{i+2}). \end{aligned}$$

We shall now prove that the second set on the right term of (2.11) is essentially empty for γ small enough. Take $x \in (x_{i+1}, x_i)$ and ε positive. Then, with $k^{(\varepsilon)} = k\chi_{(\varepsilon,\infty)}$, we have

$$\begin{aligned}|T_\varepsilon f_2(x)| &= \left|\int_{-\infty}^{x} k^{(\varepsilon)}(x-y)f_2(y)\,dy\right| = \left|\int_{-\infty}^{\bar a} k^{(\varepsilon)}(x-y)f(y)\,dy\right| \\ &\le \left|\int_{-\infty}^{\bar a}[k^{(\varepsilon)}(x-y)-k^{(\varepsilon)}(a-y)]f(y)\,dy\right| + \left|\int_{-\infty}^{\bar a} k^{(\varepsilon)}(a-y)f(y)\,dy\right| \\ &= \mathrm{I}+\mathrm{II}.\end{aligned}$$

Let us first estimate II. For $\varepsilon \ge a-\bar a$ we have

$$\left|\int_{-\infty}^{\bar a} k^{(\varepsilon)}(a-y)f(y)\,dy\right| = \left|\int_{-\infty}^{a} k^{(\varepsilon)}(a-y)f(y)\,dy\right| \le T^*f(a) \le \lambda.$$

If $\varepsilon < a-\bar a$ then

$$\left|\int_{-\infty}^{\bar a} k^{(\varepsilon)}(a-y)f(y)\,dy\right| = \left|\int_{-\infty}^{a} k^{(a-\bar a)}(a-y)f(y)\,dy\right| \le T^*f(a) \le \lambda.$$

In order to finish the proof of the lemma, we only need to show that there exists $\gamma_0 > 0$ such that for every $0 < \gamma \le \gamma_0$ and $x \in (x_{i+1}, x_i)$ with $M^-f(x) \le \gamma\lambda$ we have $\mathrm{I} \le \lambda/2$. Let $\chi_\varepsilon(t)$ denote the characteristic function of the half line (ε,∞). Then

$$\begin{aligned}\mathrm{I} &\le \int_{-\infty}^{\bar a} |k(x-y)\chi_\varepsilon(x-y)-k(a-y)\chi_\varepsilon(a-y)|\;|f(y)|\,dy \\ &\le \int_{-\infty}^{\bar a} |k(x-y)-k(a-y)|\chi_\varepsilon(a-y)\;|f(y)|\,dy \\ &+ \int_{-\infty}^{\bar a} |k(x-y)||\chi_\varepsilon(x-y)-\chi_\varepsilon(a-y)|\;|f(y)|\,dy \\ &= \mathrm{III}+\mathrm{IV}.\end{aligned}$$

Observe that $x-y > 2(x-a)$ for all $y < \bar a$. Then applying the regularity condition (1.3) we can estimate III in the following way:

$$\begin{aligned}\mathrm{III} &\le B_3\int_{-\infty}^{\bar a} \frac{(x-a)}{(x-y)^2}|f(y)|\,dy \\ &\le C(x_i-x_{i+1})\int_{-\infty}^{\bar a}\frac{|f(y)|}{(a-y)^2}\,dy \\ &\le C(x_i-x_{i+1})\sum_{j=0}^{\infty}\int_{a-(a-\bar a)2^{j+1}}^{a-(a-\bar a)2^j}\frac{|f(y)|}{(a-y)^2}\,dy \\ &\le C(x_i-x_{i+1})\sum_{j=0}^{\infty}\frac{1}{((a-\bar a)2^j)^2}\int_{a-(a-\bar a)2^{j+1}}^{\xi_0}|f(y)|\,dy \\ &\le \tilde C M^-f(\xi_0) \le \tilde C\gamma\lambda < \lambda/4,\end{aligned}$$

if $\gamma_0 < 1/(4\widetilde{C})$. On the other hand, IV can be estimated applying the size condition on k given by (1.2),

$$\begin{aligned} \mathrm{IV} &\le B \int_{-\infty}^{\bar{a}} \frac{1}{x-y} |\chi_\varepsilon(x-y) - \chi_\varepsilon(a-y)|\, |f(y)|\, dy \\ &= B \int_{(-\infty,\bar{a})\cap(a-\varepsilon,x-\varepsilon)} \frac{|f(y)|}{x-y}\, dy. \end{aligned}$$

The last term is zero for $\epsilon \le a - \bar{a}$ since the domain is empty. For $\epsilon > a - \bar{a}$ it is bounded by a constant times $M^- f(\xi_0)$. Choosing γ_0 small enough we also have $\mathrm{IV} \le \lambda/4$ and the lemma is proved. □

Proof of Theorem (2.6). We shall only prove statement (a) since the proof of (b) is analogous and (c) follows from (a) and (b) because $A_p^+ \cap A_p^- = A_p$.

Let $1 \le p < \infty$ fixed. Assume that there exists a constant C such that the inequality

$$w(\{x \in \mathbb{R} : |T_\alpha^* f(x)| > \lambda\}) \le \frac{C}{\lambda^p} \int_{\mathbb{R}} |f(x)|^p w(x)\, dx$$

holds for every $\lambda > 0$, every $\alpha > 0$ and all measurable f.

We may assume without loss of generality that $k(z) > 0$. Since the kernel is continuous in z by (1.3), there exist positive numbers δ, z_1 and z_2 with $z_1 < z_2$ such that

$$[z_1, z_2] \subset \{x : k(x) > \delta\}.$$

Let $\gamma = z_1/z_2$ and $\beta = (1-\gamma)/2$. Let $a < d$ and let us take b and c such that $a < b < c < d$ with $c - b = \gamma(d-a)$ and $d - c = b - a = \beta(d-a)$. Now choose $\alpha = z_2/(d-a) = z_1/(c-b)$. Then

$$z_1 = \alpha(c-b) \le \alpha(x-y) \le \alpha(d-a) = z_2$$

for all $x \in (c,d)$ and all $y \in (a,b)$. Therefore, if $x \in (c,d)$, $f = \sigma\chi_{(a,b)}$, $\sigma = w^{-\frac{1}{p-1}}$, we obtain

$$|T_\alpha^* f(x)| \ge \left| \int_a^b \sigma(y) k_\alpha(x-y) dy \right| \ge \alpha\delta \int_a^b \sigma(y) dy.$$

Since the maximal operators T_α^* are uniformly bounded we get

$$\int_c^d w \le \frac{C}{(\delta\alpha)^p} \left(\int_a^b \sigma \right)^{1-p} = \frac{C(d-a)^p}{\delta^p z_2^p} \left(\int_a^b \sigma \right)^{1-p},$$

and therefore, keeping in mind that δ and z_2 are constants which depend only on the kernel k, we have

$$\int_c^d w \le C(d-c) \left(\frac{b-a}{\int_a^b \sigma} \right)^{p-1} \le C(d-c) \left(M_\sigma(\sigma^{-1}\chi_{(a,d)})(a) \right)^{p-1}, \tag{2.12}$$

where

$$M_\sigma f(x) = \sup_{h,k>0} \frac{\int_{x-h}^{x+k} |f|\sigma}{\int_{x-h}^{x+k} \sigma}.$$

Now, let $x_0 = d$ and for all nonnegative integer i, let x_{i+1} be such that $x_i - x_{i+1} = \beta(x_i - a)$. Then inequality (2.12) applied to a, x_{i+1}, x_i, gives

$$\begin{aligned}\int_{x_{i+1}}^{x_i} w &\leq C(x_i - x_{i+1})\left(M_\sigma(\sigma^{-1}\chi_{(a,x_i)})(a)\right)^{p-1}\\ &\leq C(x_i - x_{i+1})\left(M_\sigma(\sigma^{-1}\chi_{(a,d)})(a)\right)^{p-1}.\end{aligned}$$

Summing up in i and dividing by $d - a$

$$\frac{1}{d-a}\int_a^d w \leq C\left(M_\sigma(\sigma^{-1}\chi_{(a,d)})(a)\right)^{p-1}.$$

Now, it follows easily from this inequality that

$$M(w\chi_{(a,d)})(a) \leq C\left(M_\sigma(\sigma^{-1}\chi_{(a,d)})(a)\right)^{p-1},$$

which is equivalent to $w \in A_p^-$ (see Theorem 2 in [MPT] for A_p^- weights). ☐

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