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ON ORTHOGONALLY EXPONENTIAL AND ORTHOGONALLY ADDITIVE MAPPINGS

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ABSTRACT. Let E be a real inner product space, $(F,+)$ an abelian σ-bounded topological group, and K a discrete subgroup of F. It is proved that (under suitable assumptions on E) the Christensen and Baire measurable orthogonally additive functions $g\colon E\to F/K$ have particular selections. In consequence, descriptions of measurable orthogonally exponential complex functionals on E are obtained.

1. Introduction

Assume the following two hypotheses:

$$(\mathrm{H}_1)\qquad E \text{ is a real inner product space with } \dim E > 1,$$

$$(\mathrm{H}_2)\qquad (F,+) \text{ is an abelian topological group and } K \text{ is a discrete subgroup of } F$$

(discrete means that there is a neighbourhood $U\subset F$ of 0 with $K\cap U=\{0\}$). We study orthogonally additive functions mapping E into the factor group F/K, i.e. functions g satisfying the condition

$$(1)\qquad g(x+y)=g(x)+g(y)\quad \text{for orthogonal } x,y\in E.$$

We show that if such a function is continuous at a point, or Christensen or Baire measurable, then, under suitable assumptions, there are continuous additive functions $a\colon R\to F$ and $A\colon E\to F$ such that $a(\|x\|^2)+A(x)\in g(x)$ for $x\in E$. In consequence, we obtain analogues, for the Baire and Christensen measurable functions, of the following theorem of K. Baron and J. Rätz.

Theorem A (see [3], p. 15)**.** *Assume* (H_1) *and* (H_2)*. Let F be continuously divisible by* 2 *(i.e. the mapping $x\to 2x$ is a homeomorphism of F onto F) and $f\colon E\to F$ be continuous at the origin and satisfying*

$$(2)\qquad f(x+y)-f(x)-f(y)\in K\quad \text{for orthogonal } x,y\in E.$$

Then there are continuous additive functions $a\colon R\to F$ and $A\colon E\to F$ such that

$$(3)\qquad f(x)-a(\|x\|^2)-A(x)\in K\quad \text{for } x\in E.$$

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We also generalize Theorem A by showing that f can be supposed continuous at any point and that the assumption of continuous divisibility by 2 can be replaced by the following weaker one:

$$(\mathrm{H}_3) \qquad 2x := x + x \neq 0 \quad \text{for } x \in F \backslash \{0\}.$$

Finally, we characterize the Baire and Christensen measurable orthogonally exponential functionals $g : E \to C$, i.e. solutions of the conditional equation

$$(4) \qquad g(x+y) = g(x)g(y) \quad \text{for orthogonal } x, y \in E.$$

The orthogonally exponential functionals $g\colon E \to C$ which are continuous at the origin or measurable on rays (i.e. for every $x \in E$ the function $t \to g(tx)$, $t \in R$, is Baire or Lebesgue measurable) have been investigated in [1] and [3]. For the information and bibliography concerning the orthogonally additive functions refer e.g. to [10] and [11].

Throughout the paper N, Z, Q, R, and C denote, as usual, the sets of positive integers, integers, rationals, reals, and complex numbers, respectively.

Given F and K satisfying (H_2), in the factor group F/K we take the factor topology, i.e. a set $U \subset F/K$ is open if the set $p^{-1}(U)$ is open in F, where $p\colon F \to F/K$ is the natural projection. If F/K is endowed with this topology, then it is a topological group and p is open and continuous.

In the sequel $\mathrm{Chr}(X)$ denotes the family of all Christensen measurable subsets of a Polish linear space X which are not Christensen zero sets (for details concerning Christensen measurability refer to [5] and [6]). Analogously, if X is a topological space, $\mathrm{Bai}(X)$ stands for the family of all subsets of X which are of the second category and with the Baire property (see e.g. [8], p. 92, and [9]). Let us recall that a function mapping a topological space X into a topological space Y is Baire measurable provided, for every open set $U \subset Y$, the set $f^{-1}(U)$ has the Baire property in X.

2. The main theorem

Let us start with the following definition and lemma.

Definition 1. We say that a topological group $(G, +)$ is σ-bounded provided, for every open neighbourhood $U \subset G$ of 0, there is a sequence $(x_n\colon n \in N) \subset G$ with

$$H = \bigcup \{U + x_n\colon n \in N\}.$$

For instance, every topological group $(G, +)$ possessing a dense countable subset is σ-bounded.

Lemma 1. *Let E be a real inner product space, $D \subset E$, and $D_0 = \{\|x\|^2\colon x \in D\}$. The following two conditions hold.*

(i) *If E is a Polish linear space and $D \in \mathrm{Chr}(E)$, then D_0 contains a subset of positive Lebesgue measure in R.*

(ii) *If $D \in \mathrm{Bai}(E)$, there is $T \in \mathrm{Bai}(R)$ with $T \subset D_0$.*

Proof. Take $e \in E$ with $\|e\| = 1$ and put $Y = \{z \in E\colon z \perp e\}$. Then Y is a linear subspace of E and $Re \oplus Y = E$.

First assume that E is a Polish space and $D \in \mathrm{Chr}(E)$. Then D has a universally measurable subset D_1 which is not a Haar zero set. Let m be the Lebesgue measure in R, $r\colon R \to E$ be given by $r(c) = ce$ for $c \in R$, and $L_k = \{c \in R\colon k-1 \leq |c| < k\}$ for $k \in N$. Define a Borel measure u on E by the formula

$$u(T) = \sum_{k=1}^{\infty} 2^{-k}[m(L_k)]^{-1} m(r^{-1}(T) \cap L_k)$$

for every Borel set $T \subset E$. It is easily seen that u extended to the family of all universally measurable subsets of E is a probability measure on E, which means that there are $b \in R$, $y \in Y$ with $u(D_1 + be + y) > 0$ (cf. [6]). Thus there is a Borel set $D_2 \subset D_1$ with $u(D_2 + be + y) > 0$. Hence $m(r^{-1}(D_2 + y)) > 0$. Further, we have $D_3 := \{c^2\colon ce - y \in D\} \supset \{c^2\colon c \in r^{-1}(D_2 + y)\}$ and

$$c^2 + \|y\|^2 = c^2\|e\|^2 + \|y\|^2 = \|ce - y\|^2 \quad \text{for } c \in R. \tag{5}$$

Consequently, D_3 contains a subset of positive Lebesgue measure and $D_3 + \|y\|^2 \subset D_0$, which implies the statement (i).

Now, suppose $D \in \mathrm{Bai}(E)$. Define a continuous functional $j\colon E \to R$ by $j(x) = \langle x, e\rangle$ for $x \in E$, where $\langle\ ,\ \rangle$ denotes the inner product in E. Then $j(ce) = c$ for $c \in R$ and $Y = \operatorname{Ker} j$. Let $g\colon E \to R \times Y$ and $h\colon R \times Y \to E$ be functions given by

$$g(z) = (j(z), s(z)) \quad \text{for } z \in E,$$
$$h(c, y) = ce + y \quad \text{for } c \in R,\ y \in Y,$$

where $s\colon E \to Y$ and $s(ce + y) = y$ for $c \in R$, $y \in Y$. Next, suppose that Y is equipped with the restriction of the inner product from E and $R \times Y$ is endowed with the product topology. Then it is easily seen that $R \times Y$ is a real topological linear space and g and h are continuous. Thus g is a homeomorphism, because $g = h^{-1}$. Hence $g(D) \in \mathrm{Bai}(R \times Y)$. Consequently there is $y \in Y$ such that

$$D_y := \{c \in R\colon (c, y) \in g(D)\} \in \mathrm{Bai}(R)$$

(see [9], p. 57) and therefore $D_1 := \{c^2 \in R\colon c \in D_y\} \in \mathrm{Bai}(R)$. Further, since (5) is valid and $\|y\| = \|-y\|$, $D_1 + \|y\|^2 \subset D_0$. This completes the proof. $\square$

Now, we are in a position to formulate and prove the following

Theorem 1. *Suppose that hypotheses* (H_1)–(H_3) *are valid and $g\colon E \to F/K$ is a function satisfying* (1). *Further, assume that one of the following three conditions holds:*

(i) *E is a Polish space, F is σ-bounded, and g is Christensen measurable;*
(ii) *E is a Baire space* (*i.e. it is of the second category*), *F is σ-bounded, and g is Baire measurable*;
(iii) *g is continuous at a point $x_0 \in E$.*

Then there are continuous additive functions $a\colon R \to F$ and $A\colon E \to F$ such that

$$a(\|x\|^2) + A(x) \in g(x) \quad \textit{for } x \in E. \tag{6}$$

Proof. Define functions $g_0, g_1, g_2\colon E \to F/K$ by $g_0(x) = g(-x)$, $g_1(x) = g(x) - g(-x) = g(x) - g_0(x)$, and $g_2(x) = g(x) + g(-x) = g(x) + g_0(x)$ for $x \in E$. It is easily seen that g_1 is odd and g_2 is even, and they are solutions of (1). Thus, by Theorems 5 and 9 in [10], g_1 is additive and there is an additive function $h\colon R \to F/K$ such that

$$g_2(x) = h(\|x\|^2) \quad \text{for } x \in E.$$

We will show that g_1 and h are continuous at the origins in E and R, respectively.

First consider the case (iii) where g is continuous at a point x_0. Then, in view of the definitions, g_1 and g_2 are continuous at x_0, too. Thus g_1 is continuous at 0, because it is additive. Take a neighbourhood $W \subset F/K$ of 0. There are neighbourhoods $U \subset F/K$ and $V \subset E$ of the respective origins such that $U - U \subset W$ and

$$g_2(V + x_0) \subset U + g_2(x_0).$$

Put $S = \{\|x\|^2 \colon x \in V + x_0\}$. It is easily seen that $\operatorname{int}(S) \neq \varnothing$ in R. Hence $S - S$ is a neighbourhood of 0 in R. To complete the proof of continuity of h at 0 it suffices to observe that

$$h(S - S) = h(S) - h(S) = g_2(V + x_0) - g_2(V + x_0) \subset U - U \subset W.$$

Now, assume that condition (i) ((ii), respectively) holds. Fix a neighbourhood $W \subset F/K$ of 0. There are open neighbourhoods $V, U \subset F/K$ of 0 such that $V = -V$, $V + V \subset U$, and $U - U \subset W$. Furthermore, since F is σ-bounded, F/K is σ-bounded, too, and consequently there exists a sequence $(x_n \colon n \in N) \subset F/K$ such that

$$F/K = \bigcup \{V + x_n \colon n \in N\}.$$

Note that

$$E = g^{-1}(F/K) \cap g_0^{-1}(F/K) = \bigcup \{g^{-1}(V + x_n) \cap g_0^{-1}(V + x_k) \colon n, k \in N\}.$$

Thus there are $n, k \in N$ such that the set

$$D := g^{-1}(V + x_n) \cap g_0^{-1}(V + x_k)$$

belongs to $\mathrm{Chr}(E)$ ($\mathrm{Bai}(E)$, resp.) and, by Lemma 1, the set D_0 contains a subset of positive Lebesgue measure in R (a subset from $\mathrm{Bai}(R)$, resp.). Hence, on account of Theorem 2 in [5] (the Difference Theorem in [8], p. 92, resp.), $0 \in \operatorname{int}(D - D)$ (in E) and $0 \in \operatorname{int}(D_0 - D_0)$ (in R). Since

$$\begin{aligned} g_1(D - D) &= g_1(D) - g_1(D) \subset (g(D) - g_0(D)) - (g(D) - g_0(D)) \\ &\subset [(V + x_n) - (V + x_k)] - [(V + x_n) - (V + x_k)] \subset U - U \subset W \end{aligned}$$

and

$$\begin{aligned} h(D_0 - D_0) &= h(D_0) - h(D_0) = g_2(D) - g_2(D) \\ &\subset (g(D) + g_0(D)) - (g(D) + g_0(D)) \subset U - U \subset W, \end{aligned}$$

this ends the proof of continuity of g_1 and h at the origins.

It results from Lemma 1 in [4] that there are functions $s_1 \colon E \to F$ and $s_2 \colon R \to F$ continuous at the origins with $s_1(x) \in g_1(x)$ for $x \in E$ and $s_2(c) \in h(c)$ for $c \in R$. Moreover, $s_1(x+y) - s_1(x) - s_1(y) \in K$ for $x, y \in E$ and $s_2(c+d) - s_2(c) - s_2(d) \in K$ for $c, d \in R$, because g_1 and h are additive. Consequently, in view of Theorem 3 in [2], there are additive and continuous functions $A_0 \colon E \to F$ and $a_0 \colon R \to F$ such that $A_0(x) \in g_1(x)$ for $x \in E$ and $a_0(c) \in h(c)$ for $c \in R$. Let $A \colon E \to F$ and $a \colon R \to F$ be given by:

$$A(x) = A_0(\tfrac{1}{2}x) \quad \text{for } x \in E,$$

$$a(c) = a_0(\tfrac{1}{2}c) \quad \text{for } c \in R.$$

Then they are continuous and additive. It remains to show that (6) holds.

To this end take a function $f\colon E \to F$ with $f(x) \in g(x)$ for $x \in E$, which means that f satisfies (2). For every $x \in E$ put $f_1(x) = f(x) - f(-x)$ and $f_2(x) = f(x) + f(-x)$. Then $f_i(x) \in g_i(x)$ for $x \in E$, $i = 1, 2$, and consequently

$$f_1(x) - A_0(x) \in K \quad \text{for } x \in E,$$

$$f_2(x) - a_0(\|x\|^2) \in K \quad \text{for } x \in E.$$

Let $k\colon E \to K$ be a function defined by the formula:

$$k(x) = f_1(x) - A_0(x) + f_2(x) - a_0(\|x\|^2) = 2f(x) - A_0(x) - a_0(\|x\|^2) \quad \text{for } x \in E.$$

Since, for every $x \in E$,

$$k(x) + k(-x) = 2[f(x) + f(-x) - a_0(\|x\|^2)] = 2[f_2(x) - a_0(\|x\|^2)] \in 2K,$$

according to Theorem 5 in [10], the function $k_0\colon E \to K/2K$, given by $k_0(x) = k(x) + 2K$ for $x \in E$, is additive. Thus $k_0(x) = 2k_0(\frac{1}{2}x) = 0$ for $x \in E$ and therefore $k(E) \subset 2K$. Whence, for every $x \in E$,

$$2[f(x) - A(x) - a(\|x\|^2)] = 2f(x) - A_0(x) - a_0(\|x\|^2) = k(x) \in 2K,$$

which jointly with (H_3) yields (6). This ends the proof. $\square$

3. Applications

Now, we present two theorems which result from Theorem 1. The first one is a generalization of Theorem 1 in [3] and contains a result concerning stability, of Hyers-Ulam type (see e.g. [7]), for orthogonally additive mappings; the second characterizes orthogonally exponential functionals.

Theorem 2. *Suppose* (H_1)–(H_3). *Let* $f\colon E \to F$ *be a function satisfying* (2). *If one of conditions* (i)–(iii) *of Theorem 1 is valid with* $g = f$, *then there exist continuous additive functions* $a\colon R \to F$ *and* $A\colon E \to F$ *such that* (3) *holds.*

Proof. Put $g = p \circ f$, where $p\colon F \to F/K$ is the natural projection. Then one of conditions (i)–(iii) of Theorem 1 is satisfied. Thus Theorem 1 implies the assertion. $\square$

Theorem 3. *Let* E *be a real inner product space with* $\dim E > 1$ *and* $h\colon E \to C$ *be a function satisfying* (4). *Suppose that one of the following three conditions is valid*:

(i) *E is a Polish space and h is Christensen measurable*;
(ii) *E is a Baire space and h is Baire measurable*;
(iii) *h is continuous at a point.*

Then either $h(x) = 0$ for $x \in E$ or

$$h(x) = \begin{cases} 0 & \text{if } x \in E \setminus \{0\}, \\ 1 & \text{if } x = 0, \end{cases}$$

or there are $c \in C$ and a continuous R-linear functional $A\colon E \to C$ such that

$$h(x) = \exp(c\|x\|^2 + A(x)) \quad \text{for } x \in E.$$

Proof. Suppose that $h(x) \neq 0$ for some $x \in E \backslash \{0\}$. Then, according to Proposition 3 in [1], $0 \notin h(E)$. Let $S = \{z \in C\colon |z| = 1\}$ and $h_0\colon E \to S$, $f\colon E \to R$, $g\colon E \to R/Z$, $T\colon S \to R/Z$ be functions given by $f(x) = \log |h(x)|$ for $x \in E$,

$$h_0(x) = \frac{h(x)}{|h(x)|} \quad \text{for } x \in E,$$

$$T(\exp 2\pi i t) = t + Z \quad \text{for } t \in [0, 1),$$

and $g = T \circ h_0$. It is easily seen that g and f satisfy the assumptions of Theorems 1 and 2, respectively, with $F = R$ and $K = Z$, and, moreover, f satisfies (1), i.e. (2) with $K = \{0\}$. Thus there are $c_1, c_2 \in R$ and continuous linear functionals $A_1, A_2\colon E \to R$ with

$$f(x) = c_1 \|x\|^2 + A_1(x) \quad \text{for } x \in E,$$

$$c_2 \|x\|^2 + A_2(x) \in g(x) \quad \text{for } x \in E.$$

Since $h(x) = h_0(x) \exp(f(x))$ for $x \in E$, setting $c = c_1 + 2\pi i c_2$ and $A = A_1 + 2\pi i A_2$ we obtain the statement. ☐

Remark. It results from Remark in [3] (on page 15) that the regularity assumptions made in Theorems 1–3 are essential.

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