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A HILBERT C^*-MODULE METHOD FOR MORITA EQUIVALENCE OF TWISTED CROSSED PRODUCTS

HUU HUNG BUI

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Abstract. We present a new proof for Morita equivalence of twisted crossed products by coactions within the abstract context of crossed products of Hilbert C^*-modules. In this context we are free from representing all C^*-algebras and Hilbert C^*-modules on Hilbert spaces.

The notion of Morita equivalence of twisted coactions was introduced in [B]. In [B, Theorem 3.3] we established conditions on twisted coactions which are sufficient to ensure Morita equivalence of the corresponding crossed product C^*-algebras. Later [ER] gave a shorter proof for this result using their results on multipliers of imprimitivity bimodules. However in the proofs of both [B] and [ER], all C^*-algebras and Hilbert C^*-modules need to be represented on Hilbert spaces.

In this paper we present a new proof for [B, Theorem 3.3] based on the notion of crossed products of Hilbert C^*-modules introduced in [B2]. Crossed products of Hilbert C^*-modules in [B2] were defined as subspaces of adjointable operators between Hilbert C^*-modules. In this abstract context, we are free from representing all C^*-algebras and Hilbert C^*-modules on Hilbert spaces as in [B] and [ER]. As a consequence, the proof here is shorter and more elegant than that of [B]. Our approach is close to the spirit of [BS], and different from [ER].

Throughout this paper G is a locally compact group and N is a closed normal amenable subgroup of G. Recall from [M, Lemma 3] that there is a surjective homomorphism Ψ from $C_r^*(G)$ into $C_r^*(G/N)$ such that $\Psi(\lambda^G(r)) = \lambda^{G/N}(q_N(r))$, where $q_N : G \to G/N$ is the quotient map, λ^G and $\lambda^{G/N}$ are the left regular representations of G and G/N. We denote by W_G the unitary operator on $L^2(G\times G)$ defined by $[W_G\xi](r,s) = \xi(r, r^{-1}s)$. If f is an element of the Fourier algebra $A(G)$, then $S_f(W_G) = M_f$. Here S_f denotes the slice map, see [LPRS, §1].

To apply [B2, Theorem 1.6] to this paper, we need to show that W_G is a regular multiplicative unitary. For any $\xi,\ \eta \in L^2(G)$, we define

$$\omega_{\eta,\xi} \;=\; \langle T\xi|\eta\rangle, \qquad \forall T \in B(L^2(G)).$$

Then for any $\omega = \omega_{\eta,\xi}$, we have

$$\langle (id\otimes\omega)(W_G)\xi'|\eta'\rangle \;= \langle M_{\omega\circ\lambda^G}\xi'|\eta'\rangle, \qquad \forall \xi',\eta' \in L^2(G).$$

It then follows that $\widehat{S}_{W_G} = C_0(G)$, and the crossed product of [B2, Proposition 1.5] is just the crossed product of [LPRS, Definition 2.4]. The unitary operator

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$\xi \longmapsto \xi$ from $L^2(G)$ onto the conjugate space $\overline{L^2(G)}$ satisfies the conditions of [BS2, Exemples 3.4.3], and hence W_G is regular.

Let (δ_D, W_D) be a twisted coaction of $(G, G/N)$ on a C^*-algebra D in the sense of [PR, Definition 2.1]. Put

$$I_{W_D} = \bigcap \{ \; ker(\pi \times \mu) \; : (\pi, \mu) \text{ is a covariant representation of } (D, G, \delta_D) \text{ which preserves } W_D \; \}.$$

The twisted crossed product $D \times_{\delta_D, W_D} G$ is the quotient $D \times_{\delta_D} G / I_{W_D}$; see [PR, Definition 2.8]. Put

$$n_D(f) \; = \; \bar{\delta}_D(S_f\{W_D\}) - [1_D \otimes M_{f \circ q_N}], \qquad \forall f \in A(G/N).$$

Recall from [B, Lemma 3.5] that I_{W_D} is the closed subspace generated by $\gamma n_D(f)\gamma'$ for all $\gamma, \gamma' \in D \times_{\delta_D} G$ and $f \in A(G/N)$.

For convenience we recall the following definition from [B, Definition 3.2].

Definition 1. Let (δ_A, W_A) and (δ_B, W_B) be twisted coactions of $(G, G/N)$ on C^*-algebras A and B. We say that (δ_A, W_A) and (δ_B, W_B) are Morita equivalent if there are an A, B-imprimitivity bimodule X and a δ_B-compatible coaction δ_X of G on X such that

(i) $\delta_X(x)\delta_X(y)^* = \overline{(\vartheta \hat{\otimes} id)} \circ \delta_A({}_A\langle x|y \rangle), \qquad \forall x, y \in X,$
(ii) $\overline{(id_X \hat{\otimes} \Psi)} \circ \delta_X(x) \; = \; W_A(x \hat{\otimes} 1)W_B^*, \qquad \forall x \in X,$

where $\vartheta : A \to \mathcal{K}(X)$ is the natural isomorphism.

Next we will give a new proof for the following result [B, Theorem 3.3].

Theorem 2. *If the twisted coactions (δ_A, W_A) and (δ_B, W_B) are Morita equivalent by means of (X, δ_X), then the twisted crossed products $A \times_{\delta_A, W_A} G$ and $B \times_{\delta_B, W_B} G$ are Morita equivalent.*

Put $E = \mathcal{K}(X)$ and $J = \mathcal{K}(X \oplus B)$. We will use the notation δ_E, δ_J, $\bar{c}_{ij}$ and $\bar{d}_{ij}$ of [B2]. Put $W_E = \overline{(\vartheta \otimes id)}(W_A)$. Then (δ_E, W_E) is a twisted coaction of $(G, G/N)$ on E, and $E \times_{\delta_E, W_E} G$ is isomorphic to $A \times_{\delta_A, W_A} G$; see [B, Lemma 3.4]. Therefore we may assume that $A = E$, $W_A = W_E$ and $\delta_A = \delta_E$.

Lemma 3. (i) *Each element $n_A(f)\delta_X(x)$ is the limit of finite sums*

$$\sum_{i=1}^{n} \delta_X(y_i) n_B(g_i), \qquad y_i \in X, \; g_i \in A(G/N).$$

(ii) *Each element $\delta_X(y)n_B(g)^*$ is the limit of finite sums*

$$\sum_{i=1}^{n} n_A(f_i)^* \delta_X(x_i), \qquad x_i \in X, \; f_i \in A(G/N).$$

Proof. (i) We write $f = g \cdot \Psi(u)$ for some $g \in A(G/N)$ and $u \in C_r^*(G)$. We shall denote $\bar{c}_{ij}(m)$ by m^c. Since $[1_A \otimes u]\delta_X(x) \in X \hat{\otimes} C_r^*(G)$, it is the limit of finite sums

$\sum_i y_i \hat{\otimes} v_i$. We compute

$$\begin{aligned}
\bar{d}_{12}\left(\bar{\delta}_A(S_f\{W_A\})\delta_X(x)\right) &= \bar{\delta}_J(\bar{c}_{11}(S_f\{W_A\}))\delta_J(x^c) \\
&= S_f\{(\delta_J \otimes id)^{-} \circ (c_{11} \otimes id)^{-}(W_A)[\delta_J(x^c) \otimes 1]\} \\
&= S_f\{(\delta_J \otimes id)^{-} \circ (c_{12} \hat{\otimes} id)^{-}(W_A[x \hat{\otimes} 1]\} \\
&= S_g\{[1_J \otimes \Psi(u)](\delta_J \otimes id)^{-} \circ (c_{12} \hat{\otimes} id)^{-}((id_X \hat{\otimes} \Psi)^{-} \circ \delta_X(x)W_B)\} \\
&= S_g\{(\delta_J \otimes \Psi) \circ (c_{12} \hat{\otimes} id)([1_A \otimes u]\delta_X(x))(\delta_J \circ c_{22} \otimes id)^{-}(W_B)\},
\end{aligned}$$

which is the limit of finite sums

$$\begin{aligned}
&\sum_{i=1}^{n} S_g\{(\delta_J \otimes \Psi) \circ (c_{12} \hat{\otimes} id)(y_i \hat{\otimes} v_i)(\delta_J \circ c_{22} \otimes id)^{-}(W_B)\} \\
&\qquad = \sum_{i=1}^{n} \delta_J(y_i^c) S_{g_i}\{(\delta_J \circ c_{22} \otimes id)^{-}(W_B)\} \\
&\qquad = \sum_{i=1}^{n} \delta_J(y_i^c) \bar{\delta}_J(\bar{c}_{22}(S_{g_i}\{W_B\})) \\
&\qquad = \bar{d}_{12}\left(\sum_{i=1}^{n} \delta_X(y_i)\bar{\delta}_B(S_{g_i}\{W_B\})\right),
\end{aligned}$$

where $g_i = g \cdot \Psi(v_i)$. Observe that $f \circ q_N = (g \circ q_N) \cdot u$ and $g_i \circ q_N = (g \circ q_N) \cdot v_i$. We compute

$$\begin{aligned}
\bar{d}_{12}\left([1_A \otimes M_{f \circ q_N}]\delta_X(x)\right) &= [1_J \otimes M_{f \circ q_N}]\delta_J(x^c) \\
&= S_{f \circ q_N}\{[1_J \otimes W_G][\delta_J(x^c) \otimes 1]\} \\
&= S_{f \circ q_N}\{(id \otimes \delta_G)^{-} \circ \delta_J(x^c)[1_J \otimes W_G]\} \\
&= S_{g \circ q_N}\{[1_J \otimes 1 \otimes u](\delta_J \otimes id)^{-} \circ \delta_J(x^c)[1_J \otimes W_G]\} \\
&= S_{g \circ q_N}\{(\delta_J \otimes id) \circ (c_{12} \hat{\otimes} id)([1_A \otimes u]\delta_X(x))[1_J \otimes W_G]\},
\end{aligned}$$

which is the limit of finite sums

$$\begin{aligned}
&\sum_{i=1}^{n} S_{g \circ q_N}\{(\delta_J \otimes id) \circ (c_{12} \hat{\otimes} id)(y_i \hat{\otimes} v_i)[1_J \otimes W_G]\} \\
&\qquad = \sum_{i=1}^{n} \delta_J(y_i^c) S_{(g \circ q_N) \cdot v_i}\{1_J \otimes W_G\} \\
&\qquad = \bar{d}_{12}\left(\sum_{i=1}^{n} \delta_X(y_i)[1_B \otimes M_{g_i \circ q_N}]\right).
\end{aligned}$$

Since $\bar{d}_{12}$ is an isometry we get the desired result.

(ii) We write $g^* = \Psi(u) \cdot f$ for some $u \in C_r^*(G)$ and $f \in A(G/N)$. Here $g^*(r) = \overline{g(r^{-1})}$. Since $\delta_X(y)[1_B \otimes u] \in X \hat{\otimes} C_r^*(G)$, it is the limit of finite sums $\sum_i x_i \hat{\otimes} v_i$. We compute

$$\begin{aligned}
\bar{d}_{12}\left(\delta_X(y)\bar{\delta}_B(S_g\{W_B\})^*\right) &= \delta_J(y^c)\bar{\delta}_J(\bar{c}_{22}(S_{g^*}\{W_B^*\})) \\
&= S_f\{(\delta_J \otimes id)^{-} \circ (c_{12} \hat{\otimes} id)^{-}([y \hat{\otimes} 1]W_B^*)[1_J \otimes \Psi(u)]\} \\
&= S_f\{(\delta_J \circ c_{11} \otimes id)^{-}(W_A^*)(\delta_J \otimes \Psi) \circ (c_{12} \hat{\otimes} id)(\delta_X(y)[1_B \otimes u])\},
\end{aligned}$$

which is the limit of finite sums

$$\sum_{i=1}^{n} S_f\{(\delta_J \circ c_{11} \otimes \bar{id})(W_A^*)(\delta_J \otimes \Psi_N) \circ (c_{12} \hat{\otimes} id)(x_i \hat{\otimes} v_i)\}$$

$$= \sum_{i=1}^{n} \bar{\delta}_J(\bar{c}_{11}(S_{f_i^*}\{W_A^*\}))\delta_J(x_i^c)$$

$$= \bar{d}_{12}\left(\sum_{i=1}^{n} \bar{\delta}_A(S_{f_i}\{W_A\})\delta_X(x_i)\right),$$

where $f_i = (\Psi(v_i)\cdot f)^*$. Observe that $(g\circ q_N)^* = u\cdot(f\circ q_N)$ and $(f_i\circ q_N)^* = v_i\cdot(f\circ q_N)$. We compute

$$\bar{d}_{12}\left(\delta_X(y)[1_B \otimes M_{g\circ q_N}]^*\right) = \delta_J(y^c)[1_J \otimes M^*_{g\circ q_N}]$$

$$= S_{(g\circ q_N)^*}\{[\delta_J(y^c) \otimes 1][1_J \otimes W_G^*]\}$$

$$= S_{(g\circ q_N)^*}\{[1_J \otimes W_G^*](id \otimes \bar{\delta_G}) \circ \delta_J(y^c)\}$$

$$= S_{f\circ q_N}\{[1_J \otimes W_G^*](\delta_J \otimes id) \circ (c_{12} \hat{\otimes} id)(\delta_X(y)[1_B \otimes u])\},$$

which is the limit of finite sums

$$\sum_{i=1}^{n} S_{f\circ q_N}\{[1_J \otimes W_G^*](\delta_J \otimes id) \circ (c_{12} \hat{\otimes} id)(x_i \hat{\otimes} v_i)\}$$

$$= \sum_{i=1}^{n} S_{v_i\cdot(f\circ q_N)}\{1_J \otimes W_G^*\}\delta_J(x_i^c)$$

$$= \bar{d}_{12}\left(\sum_{i=1}^{n} [1_A \otimes M_{f_i\circ q_N}]^* \delta_X(x_i)\right).$$

□

Note that in the Lemma 3, the proof of (ii) is very similar to that of (i). For convenience we have given both here. The arguments in the proof of Lemma 3 is also very similar to those of [B2, Proposition 1.3].

Proof of Theorem 2. Let $\mathcal{X} = X \times_{\delta_X} G$ denote the crossed product of Hilbert C^*-module X as defined in [B2, Definition 1.2]. Set $\mathcal{A} = A \times_{\delta_A} G$ and $\mathcal{B} = B \times_{\delta_B} G$. By [B2, Theorem 1.6], $\mathcal{X}$ is an $\mathcal{A},\mathcal{B}$-imprimitivity bimodule. To prove the theorem, we need to show that I_{W_A} is the ideal of $\mathcal{A}$ corresponding to the ideal I_{W_B} of $\mathcal{B}$ via $\mathcal{X}$ in the sense of [R, Theorem 3.1]. It is enough to show that $I_{W_A}\mathcal{X} = \mathcal{X}I_{W_B}$.

Let $\alpha,\ \alpha' \in \mathcal{A},\ f \in A(G/N)$ and $\xi \in \mathcal{X}$. By Lemma 3(i), $n_A(f)\alpha'\xi$ is the limit of finite sums of elements $\delta_X(y)n_B(g)\beta'$ for all $y \in X$, $g \in A(G/N)$ and $\beta' \in \mathcal{B}$. Since $\alpha\delta_X(y) \in \mathcal{X}$ for all $y \in X$, it follows that $\alpha n_A(f)\alpha'\xi \in \mathcal{X}I_{W_B}$. Hence $I_{W_A}\mathcal{X} \subset \mathcal{X}I_{W_B}$. By a similar argument and Lemma 3(ii), we can show that $\xi\beta n_B(g)^*\beta' \in I_{W_A}\mathcal{X}$ for all $\xi \in \mathcal{X}$, $\beta,\beta' \in \mathcal{B}$ and $g \in A(G/N)$. Hence $\mathcal{X}I_{W_B} \subset I_{W_A}\mathcal{X}$. □

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School of Mathematics, University of New South Wales, Sydney, New South Wales 2052, Australia

E-mail address: hung@alpha.maths.unsw.edu.au