

## ON $p$ -SUMMABLE SEQUENCES IN THE RANGE OF A VECTOR MEASURE

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ABSTRACT. Let  $p > 2$ . Among other results, we prove that a Banach space  $X$  has the property that every sequence  $(x_n) \in \ell_u^p(X)$  lies inside the range of an  $X$ -valued measure if and only if, for all sequences  $(x_n^*)$  in  $X^*$  satisfying that the operator  $x \in X \rightarrow (\langle x, x_n^* \rangle) \in \ell_1$  is 1-summing, the operator  $x \in X \rightarrow (\langle x, x_n^* \rangle) \in \ell_q$  is nuclear, being  $q$  the conjugate number for  $p$ . We also prove that, if  $X$  is an infinite-dimensional  $\mathcal{L}_p$ -space for  $1 \leq p < 2$ , then  $X$  can't have the above property for any  $s > 2$ .

### INTRODUCTION

Let  $X$  be a Banach space and let  $p > 2$ . In [Pi2] it is proved that every sequence  $(x_n)$  in  $X$  satisfying  $\sum_n |\langle x_n, x^* \rangle|^p < +\infty$  for all  $x^* \in X^*$  lies inside the range of an  $X$ -valued measure with bounded variation if and only if  $X$  is finite-dimensional. On the other hand, in [AD] the authors proved that for every Banach space and for every sequence  $(x_n) \in \ell_w^2(X)$  there is an  $X$ -valued countably additive measure whose range contains  $(x_n)$ .

The purpose of this paper is to characterize, given a real number  $p > 2$ , the Banach spaces  $X$  in which every sequence  $(x_n) \in \ell_u^p(X)$  lies inside the range of an  $X$ -valued countably additive measure.

### NOTATION

We start by explaining some basic notation used in this paper. In general, our operator and vector measure terminology and notation follow [T] and [DU]. We only consider real Banach spaces. If  $X$  is a such space,  $B_X$  will denote its closed unit ball. The phrase “range of an  $X$ -valued measure” always means a set of the form  $rg(F) = \{F(A) : A \in \Sigma\}$ , where  $\Sigma$  is a  $\sigma$ -algebra of subsets of a set  $\Omega$  and  $F : \Sigma \rightarrow X$  is countably additive. We denote by  $\|F\|$  its total semivariation

$$\|F\| = \sup\{|x^* \circ F|(\Omega) : x^* \in B_{X^*}\}.$$

Given  $p \geq 1$ ,  $\ell_w^p(X)$  will denote the vector space of all sequences  $(x_n)$  in  $X$  such that  $\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p < +\infty$  for all  $x^* \in X^*$ . It is easy to see that if  $(x_n) \in \ell_w^p(X)$

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then

$$\epsilon_p((x_n)) = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^p\right)^{\frac{1}{p}} : x^* \in B_{X^*}\right\} < +\infty$$

and  $(\ell_w^p(X), \epsilon_p)$  is itself a Banach space.

If  $\hat{x} = (x_n) \in \ell_w^p(X)$  and  $P$  is a finite subset of  $\mathbb{N}$ ,  $\hat{x}(P) = (x_n(P))$  is the sequence defined by

$$x_n(P) = \begin{cases} x_n, & \text{for } n \in P, \\ 0, & \text{for } n \notin P \end{cases}$$

for all  $n \in \mathbb{N}$ .  $\ell_u^p(X)$  will denote the subspace of  $\ell_w^p(X)$  consisting of the sequences  $\hat{x} = (x_n)$  such that the net  $(\hat{x}(P))_{P \in \mathcal{F}(\mathbb{N})}$  converges to  $(x_n)$  in  $\ell_w^p(X)$ , where  $\mathcal{F}(\mathbb{N})$  is the set of all finite subsets of  $\mathbb{N}$ . Recall that  $\ell_u^1(X)$  is formed by the unconditionally summable sequences in  $X$ . We need the following Proposition that lists some privileges that membership in  $\ell_w^p(X)$  entails.

**Proposition A.** *Let  $p > 1$  and  $X$  a Banach space. The following statements are equivalent:*

- (i)  $(x_n) \in \ell_w^p(X)$ .
- (ii) The series  $\sum_{n=1}^{\infty} \alpha_n x_n$  converges unconditionally for every sequence  $(\alpha_n) \in \ell_q$ .
- (iii) The map  $(\alpha_n) \in \ell_q \rightarrow \sum_{n=1}^{\infty} \alpha_n x_n \in X$  defines a bounded operator.

Recall that an operator  $T : X \rightarrow Y$  is said to be  $\infty$ -nuclear if there are sequences  $(x_n^*)$  in  $X^*$  and  $(y_n)$  in  $Y$  such that

$$Tx = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle y_n \quad (\forall x \in X), \quad \lim_{n \rightarrow \infty} x_n^* = 0 \quad \text{and} \quad \epsilon_1((y_n)) < +\infty.$$

A norm is then defined by taking the infimum of all admissible products

$$\left(\sup_n \|x_n^*\|\right) \epsilon_1((y_n)).$$

$\mathcal{N}_{\infty}(X, Y)$  will denote the Banach space of all  $\infty$ -nuclear operators from  $X$  into  $Y$ .

#### THE SPACES $\mathcal{R}(X)$ AND $\mathcal{R}_c(X)$

We denote by  $\mathcal{R}(X)$  the vector space of all sequences  $(x_n)$  in  $X$  so that there exists an  $X$ -valued measure  $F$  satisfying

$$(1) \quad \{x_n : n \in \mathbb{N}\} \subset \text{rg}(F).$$

If  $(x_n)$  belongs to  $\mathcal{R}(X)$ , we put  $\|(x_n)\| = \inf \|F\|$ , where the infimum is taken over all vector measures  $F$  admissible in (1). Obviously, we have

$$(2) \quad \|(x_n)\|_{\infty} \leq \|(x_n)\| \quad \text{for all } (x_n) \in \mathcal{R}(X).$$

Using a direct sum of vector measures [KK, p.35], it is easy to prove that any absolutely summable series in  $\mathcal{R}(X)$  is convergent. So,  $(\mathcal{R}(X), \|\cdot\|)$  is a Banach space.

Next we are going to consider sequences in  $X$  that lie inside the range of a vector measure with relatively compact range. We denote by  $\mathcal{R}_c(X)$  the vector space of all such sequences  $(x_n)$  in  $X$ . By [PR, Proposition 1.4], if  $(x_n)$  belongs to  $\mathcal{R}_c(X)$ , there

exists an unconditionally convergent series  $\sum_m y_m$  satisfying  $x_n \in \sum_m [-y_m, y_m]$  for all  $n$ . The set  $\sum_m [-y_m, y_m]$  is the range of a vector measure  $F$  for which

$$\|F\| \leq 2 \sup\left\{ \sum_{m=1}^{\infty} |\langle y_m, x^* \rangle| : x^* \in B_{X^*} \right\}.$$

If  $(x_n)$  belongs to  $\mathcal{R}_c(X)$ , we set

$$\|(x_n)\|_c = \inf \epsilon_1((y_m)),$$

the infimum being taken over all unconditionally convergent series  $\sum_m y_m$  such that  $(x_n)$  is contained in  $\sum_m [-y_m, y_m]$ . Obviously, we have

$$(3) \quad \|(x_n)\|_{\infty} \leq \|(x_n)\| \leq 2 \|(x_n)\|_c \text{ for all } (x_n) \in \mathcal{R}_c(X).$$

The next Proposition proves that  $\mathcal{R}_c(X)$  can be isometrically identified to  $\mathcal{N}_{\infty}(\ell_1, X)$ .

**Proposition 1.** *The Banach spaces  $\mathcal{R}_c(X)$  and  $\mathcal{N}_{\infty}(\ell_1, X)$  are isometric.*

*Proof.* If  $(x_n) \in \mathcal{R}_c(X)$ , we define an operator  $T : \ell_1 \rightarrow X$  by  $T(\alpha_n) = \sum_n \alpha_n x_n$  for all  $(\alpha_n) \in \ell_1$ . Given  $\epsilon > 0$ , choose an unconditionally summable sequence  $(y_n)$  in  $X$  so that

$$x_n = \sum_{m=1}^{\infty} \alpha_m^n y_m \text{ and } \epsilon_1((y_m)) < \epsilon + \|(x_n)\|_c,$$

where  $|\alpha_m^n| \leq 1$  for all  $n, m \in \mathbb{N}$ . Obviously,  $T\alpha = \sum_m \langle \alpha, (\alpha_m^n)_n \rangle y_m$ .

*Claim.* There exist  $(\lambda_m) \in B_{c_o}$  and  $(z_m) \in \ell_w^1(X)$ , such that  $y_m = \lambda_m z_m$  for all  $m \in \mathbb{N}$  and  $\epsilon_1((z_m)) \leq \epsilon + \epsilon_1((y_m))$ .

To prove the claim, choose  $p_o \in \mathbb{N}$  so that

$$\sum_{p \geq p_o} \frac{1}{p^2} < \epsilon.$$

Since the sequence  $(y_m)$  is unconditionally summable, we have

$$\lim_{m \rightarrow \infty} \sup\left\{ \sum_{k=m}^{\infty} |\langle y_k, x^* \rangle| : \|x^*\| \leq 1 \right\} = 0.$$

So we can determine a strictly increasing sequence of integers  $(n_p)_p$ , such that

$$\sup\left\{ \sum_{m \geq n_p} |\langle y_m, x^* \rangle| : \|x^*\| \leq 1 \right\} < \frac{1}{p^3}$$

for all  $p \in \mathbb{N}$ . Let  $(\lambda_m)$  the sequence defined by

$$\lambda_m = \begin{cases} 1, & \text{for } m \leq n_{p_o}, \\ \frac{1}{p}, & \text{for } n_p < m \leq n_{p+1}. \end{cases}$$

and put  $z_m = (\lambda_m)^{-1} y_m$  for all  $m \in \mathbb{N}$ . It is obvious that  $(\lambda_m) \in B_{c_o}$ . Hence, if we prove the inequality  $\epsilon_1((z_m)) \leq \epsilon + \epsilon_1((y_m))$ , the claim will be established. Given

$x^* \in B_{X^*}$ , we have

$$\begin{aligned} \sum_{m=1}^{\infty} |\langle z_m, x^* \rangle| &= \sum_{m=1}^{n_{p_0}} |\langle y_m, x^* \rangle| + \sum_{p=p_0}^{\infty} \sum_{k=n_p+1}^{n_{p+1}} |\langle z_k, x^* \rangle| \\ &\leq \epsilon_1((y_m)) + \sum_{p=p_0}^{\infty} p \sum_{k=n_p+1}^{n_{p+1}} |\langle y_k, x^* \rangle| \\ &\leq \epsilon_1((y_m)) + \sum_{p \geq p_0} \frac{1}{p^2} \\ &< \epsilon + \epsilon_1((y_m)). \end{aligned}$$

Now, using the claim we can write  $T$  in the form  $T\alpha = \sum_m \langle \alpha, \lambda_m(\alpha_m^n)_n \rangle z_m$ . Then  $T \in \mathcal{N}_{\infty}(\ell_1, X)$  and

$$\nu_{\infty}(T) \leq \left( \sup_m \|(\lambda_m(\alpha_m^n)_n)\|_{\infty} \right) \epsilon_1((z_m)) = \epsilon_1((z_m)) < \epsilon_1((y_m)) + \epsilon < 2\epsilon + \|(x_n)\|_c,$$

for all  $\epsilon > 0$ . Therefore, we have prove that  $\nu_{\infty}(T) \leq \|(x_n)\|_c$  for all sequences  $(x_n)$  belonging to  $R_c(X)$ .

Conversely, let  $(x_n)$  a sequence in  $X$  such that the operator

$$T : (\alpha_n) \in \ell_1 \rightarrow \sum_{n=1}^{\infty} \alpha_n x_n \in X$$

is  $\infty$ -nuclear. Given  $\epsilon > 0$ , choose a null sequence  $(\beta_m)$  in  $\ell_{\infty}$  and  $(y_m) \in \ell_w^1(X)$  so that

$$T\alpha = \sum_{m=1}^{\infty} \langle \alpha, \beta_m \rangle y_m \quad \text{for all } \alpha \in \ell_1$$

and

$$\left( \sup_m \|\beta_m\| \right) \epsilon_1((y_m)) < \nu_{\infty}(T) + \epsilon.$$

Since the sequence  $(\|\beta_m\|_{\infty} y_m)$  is unconditionally summable, it follows easily that  $(x_n)$  belongs to  $R_c(X)$  and

$$\|(x_n)\|_c < \epsilon + \nu_{\infty}(T),$$

for all  $\epsilon > 0$ . This concludes the proof.

It is well-known that with the trace duality  $\mathcal{N}_{\infty}(X, Y)^*$  and  $\Pi_1(Y, X^{**})$  can be isometrically identified, if  $X^*$  has the approximation property. In fact, for any  $S \in \Pi_1(Y, X^{**})$  the map  $\phi_S : \mathcal{N}_{\infty}(X, Y) \rightarrow \mathbb{R}$  defined by  $\phi_S(T) = \text{tr}(S \circ T)$  is a continuous linear form on  $\mathcal{N}_{\infty}(X, Y)$  and every linear form on  $\mathcal{N}_{\infty}(X, Y)$  can be obtained in this way. Accordingly,  $\mathcal{R}_c(X)^*$  and  $\Pi_1(X, \ell_1^{**})$  are isometric. The next Proposition shows that every operator  $T \in \Pi_1(X, \ell_1)$  also defines a continuous linear form on  $\mathcal{R}(X)$ .

**Proposition 2.** *Let  $X$  be a Banach space. If  $(x_n^*)$  is a sequence in  $X^*$  so that the operator  $S : x \in X \rightarrow (\langle x, x_n^* \rangle) \in \ell_1$  is 1-summing, then the linear map  $\psi_S : \mathcal{R}(X) \rightarrow \mathbb{R}$  defined by  $\psi_S((x_n)) = \sum_n \langle x_n, x_n^* \rangle$  for all  $(x_n) \in \mathcal{R}(X)$  is well-defined, continuous and  $\|\psi_S\| \leq \pi_1(S)$ .*

*Proof.* Let  $(x_n) \in \mathcal{R}(X)$ . Given  $\epsilon > 0$ , choose a vector measure  $F : \Sigma \rightarrow X$  so that

$$\{x_n : n \in \mathbb{N}\} \subset \text{rg}(F) \text{ and } \|F\| < \epsilon + \|(x_n)\|.$$

Let  $\mu$  a control measure for  $F$  and consider the integration operator

$$I : f \in L^\infty(\mu) \rightarrow \int f dF \in X.$$

The operator  $S \circ I$  is 1-summing and, hence, integral. Since  $\ell_1$  has the Radon-Nikodym property, it follows that  $S \circ I$  is nuclear and  $\pi_1(S \circ I) = \nu_1(S \circ I)$  (see [DU, p. 174]). So is  $(S \circ I)^* : \ell_\infty \rightarrow L^\infty(\mu)$ . Then

$$(4) \quad \sum_{n=1}^\infty \|(S \circ I)^*(e_n)\| < +\infty,$$

where  $(e_n)$  is the unit basis of  $c_o$ . On the other hand, we have

$$\langle (S \circ I)f, e_n \rangle = \langle f, (S \circ I)^*(e_n) \rangle.$$

Then

$$\begin{aligned} \|(S \circ I)^*(e_n)\| &= \sup\{|\langle f, (S \circ I)^*(e_n) \rangle| : \|f\|_\infty \leq 1\} \\ &= \sup\{|\int f d(x_n^* \circ F)| : \|f\|_\infty \leq 1\}. \end{aligned}$$

This shows that (4) can be written in the form

$$(5) \quad \sum_{n=1}^\infty \sup\{|\int f d(x_n^* \circ F)| : \|f\|_\infty \leq 1\} < +\infty.$$

For every  $n \in \mathbb{N}$ , we choose  $A_n \in \Sigma$  such that  $F(A_n) = x_n$ . Then, it follows from (5) that

$$(6) \quad \sum_{n=1}^\infty |\langle x_n, x_n^* \rangle| = \sum_{n=1}^\infty |\int \chi_{A_n} d(x_n^* \circ F)| < +\infty.$$

So, the linear form  $\psi_S$  is well-defined. To conclude the proof we need to show that

$$\sup\{|\sum_{n=1}^\infty \langle x_n, x_n^* \rangle| : \|(x_n)\| \leq 1\} \leq \pi_1(S).$$

If  $B$  denotes the restriction map of  $(S \circ I)^*$  to  $c_o$ , we have

$$(7) \quad \begin{aligned} |\sum_{n=1}^\infty \langle x_n, x_n^* \rangle| &\leq \sum_{n=1}^\infty \|(S \circ I)^*(e_n)\| = \nu_1(B) \leq \nu_1((S \circ I)^*) \leq \nu_1(S \circ I) \\ &= \pi_1(S \circ I). \end{aligned}$$

Since  $S \circ F$  is the representing measure of an 1-summing operator,  $S \circ I$ , it follows from [DU, p. 162] that  $\pi_1(S \circ I) = |S \circ F|$  (here,  $|S \circ F|$  denotes the total variation of  $S \circ F$ ). Finally, we can deduce from (7) that

$$|\sum_{n=1}^\infty \langle x_n, x_n^* \rangle| \leq |S \circ F| \leq \pi_1(S) \|F\| < \pi_1(S) (\epsilon + \|(x_n)\|),$$

for all  $\epsilon > 0$ . Therefore,  $\|\psi_S\| \leq \pi_1(S)$ . □

$p$ -SUMMABLE SEQUENCES IN THE RANGE OF A VECTOR MEASURE

**Proposition 3.** *Let  $X$  be a Banach space and let  $p > 2$ .  $X$  has the property that every sequence  $(x_n) \in \ell_u^p(X)$  lies inside the range of a vector measure if and only if there exists a constant  $c > 0$  so that, for all finite sets  $\{x_1, \dots, x_n\} \subset X$  satisfying  $\epsilon_p((x_i)_{i=1}^n) \leq 1$ , there is a vector measure  $F : \Sigma \rightarrow X$  satisfying*

$$\|F\| \leq c \text{ and } \{x_1, \dots, x_n\} \subset \text{rg}(F).$$

*Proof.* First, proceeding by contradiction, suppose  $\ell_u^p(X) \subset \mathcal{R}(X)$  but there isn't a such constant. Then there would exist a sequence  $(H_n)$  of finite subsets of  $X$  such that for every  $n$  the conditions

$$H_n \subset \text{rg}(F) \text{ and } \sup\left\{\left(\sum_{x \in H_n} |\langle x, x^* \rangle|^p\right)^{\frac{1}{p}} : \|x^*\| \leq 1\right\} \leq 1$$

would imply  $\|F\| \geq n^2$ . If  $H_n = \{x_1^n, \dots, x_{k(n)}^n\}$ , the sequence

$$\left\{x_1^1, \dots, x_{k(1)}^1, \frac{1}{2}x_1^2, \dots, \frac{1}{2}x_{k(2)}^2, \dots, \frac{1}{n}x_1^n, \dots, \frac{1}{n}x_{k(n)}^n, \dots\right\}$$

belongs to  $\ell_u^p(X)$ . So, there exists a vector measure  $F$  with  $(1/n)H_n \subset \text{rg}(F)$  for all  $n \in \mathbb{N}$ . Thus  $H_n \subset \text{rg}(nF)$  and this would yield  $\|F\| \geq n$  for every  $n$ , a contradiction with the fact that  $F$  has bounded semivariation since it is countably additive.

Conversely, let  $c > 0$  be a constant such that, for every finite set  $\{x_1, \dots, x_n\} \subset X$  satisfying  $\epsilon_p((x_i)_{i=1}^n) \leq 1$ , there exists a vector measure  $F$  with  $\|F\| \leq c$  and whose range contains  $\{x_1, \dots, x_n\}$ . Then, the natural map from  $(X^{(\mathbb{N})}, \epsilon_p)$  to  $(\mathcal{R}(X), \|\cdot\|)$  is linear and continuous. Therefore, it has a unique continuous linear extension to  $\ell_u^p(X)$ .

Now we are ready to face our main result. □

**Theorem 4.** *Let  $X$  be a Banach space and let  $p > 2$ . The following are equivalent:*

- (i) *Every sequence  $(x_n) \in \ell_u^p(X)$  lies inside the range of a vector measure.*
- (ii) *For all sequence  $(x_n^*)$  in  $X^*$  satisfying that  $x \in X \rightarrow (\langle x, x_n^* \rangle) \in \ell_1$  is 1-summing, the operator  $x \in X \rightarrow (\langle x, x_n^* \rangle) \in \ell_q$  is nuclear.*
- (iii) *There exists a constant  $c > 0$  such that, for all  $n \in \mathbb{N}$ , for all  $\{x_1, \dots, x_n\}$  in  $X$  and all  $\{x_1^*, \dots, x_n^*\}$  in  $X^*$ , we have*

$$\sum_{i=1}^n |\langle x_i, x_i^* \rangle| \leq c \pi_1\left(\sum_{i=1}^n x_i^* \otimes e_i : X \rightarrow \ell_1^n\right) \epsilon_p((x_i)_{i=1}^n).$$

*Proof.* (i)  $\Rightarrow$  (ii) We have seen in the proof of Proposition 3 that the natural map  $\ell_u^p(X) \rightarrow \mathcal{R}(X)$  is continuous. So, given a sequence  $(x_n^*)$  in  $X^*$  such that the map

$$x \in X \rightarrow (\langle x, x_n^* \rangle) \in \ell_1$$

is 1-summing, it follows from Proposition 2 that the linear form  $\phi$  on  $\ell_u^p(X)$  defined by  $\phi(x_n) = \sum_n \langle x_n, x_n^* \rangle$  is continuous. Then the linear map  $x \in X \rightarrow (\langle x, x_n^* \rangle) \in \ell_q$  is integral [DU, p. 232]. Now recall that nuclear and integral operators into a reflexive space are the same [DU].

(ii)  $\Rightarrow$  (iii) The linear map

$$T \in \Pi_1(X, \ell_1) \rightarrow I_{1q} \circ T \in \mathcal{N}(X, \ell_q)$$

has closed graph (here  $I_{1q}$  is the inclusion map from  $\ell_1$  into  $\ell_q$ ). So, it is continuous. Then there is a constant  $c > 0$  such that

$$(8) \quad \nu_1(I_{1q}^n \circ T) \leq c \pi_1(T)$$

for all  $n \in \mathbb{N}$  and all  $T \in \mathcal{L}(X, \ell_1^n)$ , being  $I_{1q}^n : \ell_1^n \rightarrow \ell_q^n$  the identity map. Now, given  $\{x_1, \dots, x_n\}$  in  $X$  and  $\{x_1^*, \dots, x_n^*\}$  in  $X^*$ , we consider the operators  $u : X \rightarrow \ell_q^n$  and  $v : \ell_q^n \rightarrow X$  defined by

$$u(x) = (\langle x, x_i^* \rangle)_{i=1}^n \quad \text{and} \quad v((\alpha_i)) = \sum_{i=1}^n \alpha_i x_i.$$

Since  $\text{tr}(u \circ v) = \sum_{i=1}^n \langle x_i, x_i^* \rangle$ , we have

$$\left| \sum_{i=1}^n \langle x_i, x_i^* \rangle \right| \leq \nu_1^o(u \circ v) = \nu_1(u \circ v) \leq \nu_1(u) \|v\|,$$

being  $\nu_1^o$  the finite-nuclear norm [P]. Then, by (8) we obtain

$$\left| \sum_{i=1}^n \langle x_i, x_i^* \rangle \right| \leq \pi_1 \left( \sum_{i=1}^n x_i^* \otimes e_i : X \rightarrow \ell_1^n \right) \|v\|.$$

This yields (iii) because of the equality  $\|v\| = \epsilon_p((x_i)_{i=1}^n)$ .

(iii)  $\Rightarrow$  (i) For every  $n \in \mathbb{N}$ , we define a linear map

$$U_n : (X^n, \epsilon_p) \rightarrow \mathcal{N}_\infty(\ell_1^n, X)$$

in the following way: Given  $\hat{x} = (x_i)_1^n \in X^n$ , we denote by  $T_{\hat{x}}$  the operator from  $\ell_1^n$  into  $X$  defined by  $T_{\hat{x}}((\alpha_i)_1^n) = \sum_{i=1}^n \alpha_i x_i$ . Then we put  $U_n(\hat{x}) = T_{\hat{x}}$ . To show that  $U_n$  is continuous, we are going to compute  $\nu_\infty(T_{\hat{x}})$ :

$$\begin{aligned} \nu_\infty(T_{\hat{x}}) &= \sup\{|\langle T_{\hat{x}}, S \rangle| : \pi_1(S) \leq 1, S \in \Pi_1(X, \ell_1^n)\} \\ &= \sup\{|\text{tr}(S \circ T_{\hat{x}})| : \pi_1(S) \leq 1, S \in \Pi_1(X, \ell_1^n)\} \\ &= \sup\left\{ \left| \sum_{i=1}^n \langle x_i, S^*(e_i^*) \rangle \right| : \pi_1(S) \leq 1, S \in \Pi_1(X, \ell_1^n) \right\}, \end{aligned}$$

where  $(e_i^*)_1^n$  is the unit basis of  $\ell_\infty^n$ . By (iii), we obtain

$$\nu_\infty(T_{\hat{x}}) \leq c \epsilon_p((x_i)_1^n).$$

So, we have

$$(9) \quad \|U_n\| \leq c \quad \text{for every } n \in \mathbb{N}.$$

Now we consider the linear map

$$U : (X^{(\mathbb{N})}, \epsilon_p) \rightarrow \mathcal{N}_\infty(\ell_1, X)$$

defined by  $U(\hat{x}) = T_{\hat{x}}$  with  $T_{\hat{x}}((\alpha_n)) = \sum_n \alpha_n x_n$ , for all  $\hat{x} = (x_n) \in X^{(\mathbb{N})}$  and all  $(\alpha_n) \in \ell_1$ . From (9) it follows that  $\|U\| \leq c$ . Finally, Proposition 1 implies that  $\ell_u^p(X)$  is contained in  $\mathcal{R}_c(X)$ .

Using the methods introduced in [Pi2], we can prove that an operator  $T : X \rightarrow Y$  takes every sequence  $(x_n) \in \ell_u^p(X)$  into a sequence  $(Tx_n)$  lying in the range of an  $X$ -valued measure with bounded variation if and only if  $T^* : Y^* \rightarrow X^*$  is  $(1, 1, p)$ -summing. The next Proposition shows the relationship between these ideas and our problem.  $\square$

**Proposition 5.**  $\ell_u^p(X)$  is contained in  $\mathcal{R}(X)$  if and only if  $T^* : Y^* \rightarrow X^*$  is  $(1, 1, p)$ -summing for all Banach space  $Y$  and for all  $T \in \Pi_1(X, Y)$ .

*Proof.* Suppose  $X$  is a Banach space such that  $\ell_u^p(X) \subset \mathcal{R}(X)$ . If  $T \in \Pi_1(X, Y)$ , then  $T$  takes every sequence  $(x_n) \in \ell_u^p(X)$  into a sequence  $(Tx_n)$  lying in the range of a vector measure of bounded variation. So,  $T^* : Y^* \rightarrow X^*$  is  $(1, 1, p)$ -summing.

Conversely, let  $X$  be a Banach space with the property that, for all  $T \in \Pi_1(X, \ell_1)$ , the operator  $T^*$  is  $(1, 1, p)$ -summing. Take a sequence  $(x_n^*)$  in  $X^*$  such that  $T : x \in X \rightarrow (\langle x, x_n^* \rangle) \in \ell_1$  is 1-summing. By hypothesis,  $T^*$  is  $(1, 1, p)$ -summing. Now an appeal to [Pi2, Theorem 1] yields the nuclearity of the operator

$$x^{**} \in X^{**} \rightarrow (\langle x^{**}, T^* e_n^* \rangle) \in \ell_q.$$

Then  $x \in X \rightarrow (\langle x, x_n^* \rangle) \in \ell_q$  is also nuclear. Now Theorem 4 tells us that  $\ell_u^p(X) \subset \mathcal{R}(X)$ . □

EXAMPLES

If  $X$  is a Banach space,  $s(X)$  will denote the set of all real numbers  $s > 2$  such that every sequence  $(x_n) \in \ell_u^s(X)$  lies inside the range of an  $X$ -valued measure. Obviously,  $s(X)$  is an interval whose bounds are 2 and  $\sup(s(X))$ . In this section we find out the form of the set  $s(X)$  for any  $\mathcal{L}_p$ -space  $X$  ( $1 \leq p \leq +\infty$ ).

a) *If  $X$  is an  $\mathcal{L}_p$ -space for  $p \geq 2$ , then  $s(X) = (2, +\infty)$ .*

By [LR, Theorem III],  $X^*$  is an  $\mathcal{L}_q$ -space. So  $X^*$  is isomorphic to a subspace of an  $L_1(\mu)$ -space for some measure  $\mu$  [LP, Corollary 7.2]. Hence, every null sequence in  $X$  lies inside the range of an  $X$ -valued measure [PR].

Following [Ps2] we will say that a Banach space  $X$  satisfies Grothendieck's Theorem (in short,  $X$  is a G. T. space) if  $\mathcal{L}(X, \ell_2) = \Pi_1(X, \ell_2)$ . For some time the  $\mathcal{L}_1$ -spaces remained the only known G. T. spaces. We have just seen that  $s(X) = (2, +\infty)$  for every Banach space  $X$  whose dual  $X^*$  is an  $\mathcal{L}_1$ -space. Nevertheless, there are spaces for which  $s(X) = \emptyset$  and  $X^*$  is a G. T. space. Next we will prove this assertion.

b) *If  $X$  is an infinite-dimensional Banach space of cotype 2 and  $X^*$  is a G. T. space, then  $s(X) = \emptyset$ .*

Since  $X$  is a space of cotype 2 we have the identity  $\Pi_1(X, \ell_1) = \Pi_2(X, \ell_1)$  [Ps2, Theorem 5.16]. By [Ps2, Proposition 6.2]  $\Pi_2(X, \ell_1) = \mathcal{L}(X, \ell_1)$ . So, it follows that

$$(10) \quad \pi_1(u) \leq c\|u\| \quad \text{for all } u \in \mathcal{L}(X, \ell_1)$$

and for some constant  $c > 0$ . Proceeding by contradiction, suppose  $p \in s(X)$ . Then there exists a constant  $c' > 0$  so that

$$(11) \quad \sum_{i=1}^n |\langle x_i, x_i^* \rangle| \leq c' \pi_1 \left( \sum_i^n x_i^* \otimes e_i : X \rightarrow \ell_1^n \right) \epsilon_p((x_i)_{i=1}^n)$$

for all  $\{x_i\}_{i=1}^n$  in  $X$  and all  $\{x_i^*\}_{i=1}^n$  in  $X^*$ . (10) and (11) yield

$$(12) \quad \sum_{i=1}^n |\langle x_i, x_i^* \rangle| \leq c' c \epsilon_p((x_i)_{i=1}^n) \epsilon_1((x_i^*)_{i=1}^n)$$

for all  $\{x_i\}_{i=1}^n$  in  $X$  and all  $\{x_i^*\}_{i=1}^n$  in  $X^*$ . This proves that the identity map  $I_X$  is  $(1, p, 1)$ -summing [P]. Hence  $I_{X^*}$  is  $(1, 1, p)$ -summing [P, Theorem 17.1.5]. Since  $p > 2$ , from [Pi2] it follows that  $X^*$  is finite-dimensional, a contradiction which completes the proof.

In [Ps1, Theorem 3.2], it is proved that any Banach space  $E$  of cotype 2 can be imbedded isometrically into a Banach space  $X$  such that  $X$  and  $X^*$  are both of cotype 2 and both verify Grothendieck's Theorem. So, if  $X$  is the Banach space associated to  $E = \ell_2$  in [Ps1, Theorem 3.2], for every  $p > 2$  there are  $p$ -summable sequences in  $X$  that do not lie inside the range of a  $X$ -valued vector measure. Nevertheless, the closed unit ball of  $\ell_2$  is the range of a vector measure [AD].

c)  $s(\ell_1) = \emptyset$ .

In [AD] it is proved that the sequence  $(e_n/\sqrt{n})$  does not lie inside the range of a measure. Nevertheless,  $(e_n/\sqrt{n})$  belongs to  $\ell_u^s(\ell_1)$  for any  $s > 2$ .

d)  $s(\ell_p) = \emptyset$  for any  $1 < p < 2$ .

Given  $s \in (2, q)$ , we choose  $\alpha \in (1/s - 1/q, 1/2 - 1/q)$ . We are going to prove that the sequence  $(e_n/n^\alpha)$  belongs to  $\ell_u^s(\ell_p)$  but it isn't in the range of a measure. If  $x^* = (x_n^*) \in \ell_q$  with  $\|x^*\|_q \leq 1$ , we have for all  $m \in \mathbb{N}$ :

$$\sum_{n=m}^{\infty} |\langle \frac{e_n}{n^\alpha}, x^* \rangle|^s = \sum_{n=m}^{\infty} \frac{1}{n^{\alpha s}} |x_n^*|^s \leq \left( \sum_{n=m}^{\infty} \frac{1}{n^{\frac{\alpha s q}{q-s}}} \right)^{\frac{q-s}{q}} \left( \sum_{n=m}^{\infty} |x_n^*|^q \right)^{\frac{s}{q}}$$

where we have applied Holder's inequality. Then

$$\sup \left\{ \sum_{n=m}^{\infty} |\langle \frac{e_n}{n^\alpha}, x^* \rangle|^s : \|x^*\|_q \leq 1 \right\} \leq \left( \sum_{n=m}^{\infty} \frac{1}{n^{\frac{\alpha s q}{q-s}}} \right)^{\frac{q-s}{q}}$$

for all  $m \in \mathbb{N}$ . Since  $\alpha s q / (q - s) > 1$ , it follows that  $(e_n/n^\alpha)$  is plainly in  $\ell_u^s(\ell_p)$ . The sequence  $(e_n/n^\alpha)$  is an unconditional basis of  $\ell_p$ ; therefore, to prove that the sequence  $(e_n/n^\alpha)$  is not in the range of a measure it suffices to show that it does not belong to  $\ell_w^2(\ell_p)$  [AD, Theorem 5]. For this, choose  $\beta > (1/q)$  and consider the scalar sequence  $x^* = (1/n^\beta)$ . It is plain that  $x^* \in \ell_q$  and that viewed as a member of  $(\ell_p)^*$ ,  $x^*$  satisfies:

$$\sum_n |\langle \frac{e_n}{n^\alpha}, x^* \rangle|^2 = \sum_{n=1}^{\infty} \frac{1}{n^{2(\alpha+\beta)}}.$$

Therefore, if we choose  $\beta \in (1/q, 1/2 - \alpha)$ , the last series is divergent and we deduce that  $s \notin s(\ell_p)$ .

e) If  $X$  is an infinite-dimensional  $\mathcal{L}_p$ -space for  $1 \leq p < 2$ , then  $s(X) = \emptyset$ .

By [LP, Proposition 7.3],  $X$  has a complemented subspace  $H$  isomorphic to  $\ell_p$ . Then  $s(X) \subset s(H) = s(\ell_p) = \emptyset$ .

In view of these results and [PR] a question arises: *Are there Banach spaces  $X$  such that  $s(X) \neq \emptyset$  but  $X^*$  is not isomorphic to a subspace of an  $L_1(\mu)$ -space for some measure  $\mu$ ?*

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