

BASIC DIFFERENTIAL FORMS FOR ACTIONS OF LIE GROUPS. II

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ABSTRACT. The assumption in the main result of *Basic differential forms for actions of Lie groups* (Proc. Amer. Math. Soc. **124** (1996), 1633–1642) is removed.

Let G be a Lie group which acts isometrically on a Riemannian manifold M . A section of the Riemannian G -manifold M is a closed submanifold Σ which meets each orbit orthogonally. In this situation the trace on Σ of the G -action is a discrete group action by the generalized Weyl group $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$, where $N_G(\Sigma) := \{g \in G : g.\Sigma = \Sigma\}$ and $Z_G(\Sigma) := \{g \in G : g.s = s \text{ for all } s \in \Sigma\}$. A differential form $\varphi \in \Omega^p(M)$ is called G -invariant if $g^*\varphi = \varphi$ for all $g \in G$ and horizontal if φ kills each vector tangent to a G -orbit. We denote by $\Omega_{\text{hor}}^p(M)^G$ the space of all horizontal G -invariant p -forms on M which are also called *basic forms*.

In the paper [2] it was shown that for a proper isometric action of a Lie group G on a smooth Riemannian manifold M admitting a section Σ the restriction of differential forms induces an isomorphism

$$\Omega_{\text{hor}}^p(M)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)}$$

between the space of horizontal G -invariant differential forms on M and the space of all differential forms on Σ which are invariant under the action of the generalized Weyl group $W(\Sigma)$ of the section Σ , under the following assumption:

For each $x \in \Sigma$ the slice representation $G_x \rightarrow O(T_x(G.x)^\perp)$ has a generalized Weyl group which is a reflection group.

In this paper we will show that this result holds in general, without any assumption. Notation is as in [2], which is used throughout. For more information on G -manifolds with sections see the seminal paper [3].

1. Polar representations. Let G be a compact Lie group and let $\rho : G \rightarrow GL(V)$ be an orthogonal representation in a finite dimensional real vector space V which admits a section Σ . Then the section turns out to be a linear subspace and the representation is called a *polar representation*, following Dadok [1], who gave a complete classification of all polar representations of connected Lie groups.

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Theorem. *Let $\rho : G \rightarrow O(V)$ be a polar orthogonal representation of a compact Lie group G , with section Σ and generalized Weyl group $W = W(\Sigma)$. Let $B \subset V$ be an open ball centered at 0 .*

Then the restriction of differential forms induces an isomorphism

$$\Omega_{hor}^p(B)^G \xrightarrow{\cong} \Omega^p(\Sigma \cap B)^{W(\Sigma)}.$$

Proof. We only treat the case $B = V$. The restriction to an open ball can be proved as in [2], 3.8. Let $i : \Sigma \rightarrow V$ be the embedding. It is easy to see (and proved in [2], 2.4) that the restriction $i^* : \Omega_{hor}^p(V)^G \rightarrow \Omega^p(\Sigma)^{W(G)}$ is injective, so it remains to prove surjectivity. Let G_0 be the connected component of G . From [1], lemma 1 one concludes:

A subspace Σ of V is a section for G if and only if it is a section for G_0 . Thus ρ is a polar representation for G if and only if it is a polar representation for G_0 .

The generalized Weyl groups of Σ with respect to G and to G_0 are related by

$$W(G_0) = N_{G_0}(\Sigma)/Z_{G_0}(\Sigma) \subset W(G) = N_G(\Sigma)/Z_G(\Sigma),$$

since $Z_G(\Sigma) \cap N_{G_0}(\Sigma) = Z_{G_0}(\Sigma)$.

Let $\omega \in \Omega^p(\Sigma)^{W(G)} \subset \Omega^p(\Sigma)^{W(G_0)}$. Since G_0 is connected the generalized Weyl group $W(G_0)$ is generated by reflections (a Coxeter group) by [1], remark after proposition 6. Thus we may apply [2], theorem 3.7, which asserts that then

$$i^* : \Omega_{hor}^p(V)^{G_0} \xrightarrow{\cong} \Omega^p(\Sigma)^{W(G_0)}$$

is an isomorphism, and we get $\varphi \in \Omega_{hor}^p(V)^{G_0}$ with $i^*\varphi = \omega$. Let us consider

$$\psi := \int_G g^*\varphi dg \in \Omega_{hor}^p(V)^G,$$

where dg denotes Haar measure on G . In order to show that $i^*\psi = \omega$ it suffices to check that $i^*g^*\varphi = \omega$ for each $g \in G$. Now $g(\Sigma)$ is again a section of G , thus also of G_0 . Since any two sections are related by an element of the group, there exists $h \in G_0$ such that $hg(\Sigma) = \Sigma$. Then $hg \in N_G(\Sigma)$ and we denote by $[hg]$ the coset in $W(G)$, and we may compute as follows:

$$\begin{aligned} (i^*g^*\varphi)_x &= (g^*\varphi)_x \cdot \Lambda^p T i = \varphi_{g(x)} \cdot \Lambda^p T g \cdot \Lambda^p T i \\ &= (h^*\varphi)_{g(x)} \cdot \Lambda^p T g \cdot \Lambda^p T i, \quad \text{since } \varphi \in \Omega_{hor}^p(M)^{G_0} \\ &= \varphi_{hg(x)} \cdot \Lambda^p T(hg) \cdot \Lambda^p T i = \varphi_{i[hg](x)} \cdot \Lambda^p T i \cdot \Lambda^p T([hg]) \\ &= (i^*\varphi)_{[hg](x)} \cdot \Lambda^p T([hg]) \\ &= \omega_{[hg](x)} \cdot \Lambda^p T([hg]) = [hg]^*\omega = \omega. \quad \square \end{aligned}$$

2. Theorem. *Let $M \times G \rightarrow M$ be a proper isometric right action of a Lie group G on a smooth Riemannian manifold M , which admits a section Σ .*

Then the restriction of differential forms induces an isomorphism

$$\Omega_{hor}^p(M)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)}$$

between the space of horizontal G -invariant differential forms on M and the space of all differential forms on Σ which are invariant under the action of the generalized Weyl group $W(\Sigma)$ of the section Σ .

This is the Main Theorem 2.4 of [2], without the assumption made there.

Proof. Injectivity is proved in [2], 2.4, without using the assumption. Surjectivity can be proved as in [2], section 4, where one replaces the use of [2], 3.8, by Theorem 1 above. \square

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