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BASIC DIFFERENTIAL FORMS FOR ACTIONS OF LIE GROUPS. II

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Abstract. The assumption in the main result of *Basic differential forms for actions of Lie groups* (Proc. Amer. Math. Soc. **124** (1996), 1633–1642) is removed.

Let G be a Lie group which acts isometrically on a Riemannian manifold M. A section of the Riemannian G-manifold M is a closed submanifold Σ which meets each orbit orthogonally. In this situation the trace on Σ of the G-action is a discrete group action by the generalized Weyl group $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$, where $N_G(\Sigma) := \{g \in G : g.\Sigma = \Sigma\}$ and $Z_G(\Sigma) := \{g \in G : g.s = s \text{ for all } s \in \Sigma\}$. A differential form $\varphi \in \Omega^p(M)$ is called *G-invariant* if $g^*\varphi = \varphi$ for all $g \in G$ and *horizontal* if φ kills each vector tangent to a G-orbit. We denote by $\Omega^p_{\text{hor}}(M)^G$ the space of all horizontal G-invariant p-forms on M which are also called *basic forms.*

In the paper [2] it was shown that for a proper isometric action of a Lie group G on a smooth Riemannian manifold M admitting a section Σ the restriction of differential forms induces an isomorphism

$$\Omega^p_{\text{hor}}(M)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)}$$

between the space of horizontal G-invariant differential forms on M and the space of all differential forms on Σ which are invariant under the action of the generalized Weyl group $W(\Sigma)$ of the section Σ, under the following assumption:

> For each $x \in \Sigma$ the slice representation $G_x \to O(T_x(G.x)^\perp)$ has a generalized Weyl group which is a reflection group.

In this paper we will show that this result holds in general, without any assumption. Notation is as in [2], which is used throughout. For more information on G-manifolds with sections see the seminal paper [3].

1. Polar representations. Let G be a compact Lie group and let $\rho : G \to GL(V)$ be an orthogonal representation in a finite dimensional real vector space V which admits a section Σ. Then the section turns out to be a linear subspace and the representation is called a *polar representation*, following Dadok [1], who gave a complete classification of all polar representations of connected Lie groups.

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Theorem. *Let $\rho : G \to O(V)$ be a polar orthogonal representation of a compact Lie group G, with section Σ and generalized Weyl group $W = W(\Sigma)$. Let $B \subset V$ be an open ball centered at 0.*

Then the restriction of differential forms induces an isomorphism

$$\Omega^p_{hor}(B)^G \xrightarrow{\cong} \Omega^p(\Sigma \cap B)^{W(\Sigma)}.$$

Proof. We only treat the case $B = V$. The restriction to an open ball can be proved as in [2], 3.8. Let $i : \Sigma \to V$ be the embedding. It is easy to see (and proved in [2], 2.4) that the restriction $i^* : \Omega^p_{\mathrm{hor}}(V)^G \to \Omega^p(\Sigma)^{W(G)}$ is injective, so it remains to prove surjectivity. Let G_0 be the connected component of G. From [1], lemma 1 one concludes:

> *A subspace Σ of V is a section for G if and only if it is a section for G_0. Thus ρ is a polar representation for G if and only if it is a polar representation for G_0.*

The generalized Weyl groups of Σ with respect to G and to G_0 are related by

$$W(G_0) = N_{G_0}(\Sigma)/Z_{G_0}(\Sigma) \subset W(G) = N_G(\Sigma)/Z_G(\Sigma),$$

since $Z_G(\Sigma) \cap N_{G_0}(\Sigma) = Z_{G_0}(\Sigma)$.

Let $\omega \in \Omega^p(\Sigma)^{W(G)} \subset \Omega^p(\Sigma)^{W(G_0)}$. Since G_0 is connected the generalized Weyl group $W(G_0)$ is generated by reflections (a Coxeter group) by [1], remark after proposition 6. Thus we may apply [2], theorem 3.7, which asserts that then

$$i^* : \Omega^p_{\mathrm{hor}}(V)^{G_0} \xrightarrow{\cong} \Omega^p(\Sigma)^{W(G_0)}$$

is an isomorphism, and we get $\varphi \in \Omega^p_{\mathrm{hor}}(V)^{G_0}$ with $i^*\varphi = \omega$. Let us consider

$$\psi := \int_G g^*\varphi \, dg \in \Omega^p_{\mathrm{hor}}(V)^G,$$

where dg denotes Haar measure on G. In order to show that $i^*\psi = \omega$ it suffices to check that $i^*g^*\varphi = \omega$ for each $g \in G$. Now $g(\Sigma)$ is again a section of G, thus also of G_0. Since any two sections are related by an element of the group, there exists $h \in G_0$ such that $hg(\Sigma) = \Sigma$. Then $hg \in N_G(\Sigma)$ and we denote by $[hg]$ the coset in $W(G)$, and we may compute as follows:

$$\begin{aligned}
(i^*g^*\varphi)_x &= (g^*\varphi)_x.\Lambda^pTi = \varphi_{g(x)}.\Lambda^pTg.\Lambda^pTi \\
&= (h^*\varphi)_{g(x)}.\Lambda^pTg.\Lambda^pTi, \qquad \text{since } \varphi \in \Omega^p_{\mathrm{hor}}(M)^{G_0} \\
&= \varphi_{hg(x)}.\Lambda^pT(hg).\Lambda^pTi = \varphi_{i[hg](x)}.\Lambda^pTi.\Lambda^pT([hg]) \\
&= (i^*\varphi)_{[hg](x)}.\Lambda^pT([hg]) \\
&= \omega_{[hg](x)}.\Lambda^pT([hg]) = [hg]^*\omega = \omega. \qquad \square
\end{aligned}$$

2. Theorem. *Let $M \times G \to M$ be a proper isometric right action of a Lie group G on a smooth Riemannian manifold M, which admits a section Σ.*

Then the restriction of differential forms induces an isomorphism

$$\Omega^p_{hor}(M)^G \xrightarrow{\cong} \Omega^p(\Sigma)^{W(\Sigma)}$$

between the space of horizontal G-invariant differential forms on M and the space of all differential forms on Σ which are invariant under the action of the generalized Weyl group $W(\Sigma)$ of the section Σ.

This is the Main Theorem 2.4 of [2], without the assumption made there.

Proof. Injectivity is proved in [2], 2.4, without using the assumption. Surjectivity can be proved as in [2], section 4, where one replaces the use of [2], 3.8, by Theorem 1 above. □

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