

## INTERSECTION OF ESSENTIAL IDEALS IN $C(X)$

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**ABSTRACT.** The infinite intersection of essential ideals in any ring may not be an essential ideal, this intersection may even be zero. By the topological characterization of the socle by Karamzadeh and Rostami (Proc. Amer. Math. Soc. **93** (1985), 179–184), and the topological characterization of essential ideals in Proposition 2.1, it is easy to see that every intersection of essential ideals of  $C(X)$  is an essential ideal if and only if the set of isolated points of  $X$  is dense in  $X$ . Motivated by this result in  $C(X)$ , we study the essentiality of the intersection of essential ideals for topological spaces which may have no isolated points. In particular, some important ideals  $C_K(X)$  and  $C_\infty(X)$ , which are the intersection of essential ideals, are studied further and their essentiality is characterized. Finally a question raised by Karamzadeh and Rostami, namely when the socle of  $C(X)$  and the ideal of  $C_K(X)$  coincide, is answered.

### 1. INTRODUCTION

In this paper, we denote by  $C(X)$  the ring of real-valued, continuous functions on a completely regular space  $X$ , and the reader is referred to [3] for undefined terms and notations.

A non-zero ideal in a commutative ring is said to be essential if it intersects every non-zero ideal non-trivially, and the intersection of all essential ideals, or the sum of all minimal ideals, is called the socle (see [5]). We denote the socle of  $C(X)$  by  $C_F(X)$ ; it is characterized in [4] as the set of all functions which vanish everywhere except on a finite number of points of  $X$ .

The familiar ideals  $C_K(X)$ , the set of functions with compact support, and  $C_\infty(X)$ , the set of functions vanishing at infinity, are also ideals which can be represented by the intersection of some essential ideals. In fact, these ideals are the intersection of free ideals (see [3, 7E]), and by Proposition 2.1, free ideals are essential ideals. We will show that  $C_K(X)$  and  $C_\infty(X)$  are essential ideals in  $C(X)$  if and only if every open subset of  $X$  contains an open set with compact closure. It is well known that for a locally compact, non-compact space  $X$ ,  $C_K(X) = C_\infty(X)$  if and only if every  $\sigma$ -compact set of  $X$  is contained in a compact set in  $X$  (see [3, 7G]). We observe that  $C_F(X) = C_K(X)$  if and only if every compact subset of  $X$  has a finite interior, and we also characterize the spaces  $X$  for which  $C_F(X) = C_\infty(X)$ .

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## 2. ESSENTIAL IDEALS AND THEIR INTERSECTION

The following proposition which topologically characterizes essential ideals is also proved in [1].

**Proposition 2.1.** *A non-zero ideal  $E$  in  $C(X)$  is an essential ideal if and only if  $\cap Z[E]$  is nowhere dense in  $X$ .*

*Proof.* Let  $E$  be an essential ideal and  $\text{int} \cap Z\{E\} \neq \emptyset$ . If  $x \in \text{int} \cap Z[E]$ , then by the complete regularity of  $X$ , there is some  $g \in C(X)$  for which  $g(x) = 1$  and  $g(X \setminus \text{int} \cap Z[E]) = 0$ . Now  $E \cap (g) = 0$ , for if  $f \in E \cap (g)$ , then  $X = (\cap Z[E]) \cup Z(g) \subseteq Z(f)$ , i.e.,  $f = 0$ .

Conversely, suppose that  $\text{int} \cap Z[E] = \emptyset$ . Then  $X \setminus \cap Z[E]$  is open and dense in  $X$ . Let  $I$  be any non-zero ideal and  $0 \neq g \in I$ . Then  $X \setminus Z(g)$  is a non-empty open set, so  $[X \setminus Z(g)] \cap (X \setminus \cap Z[E]) \neq \emptyset$ . This implies there is a function  $f \in E$  for which  $[X \setminus Z(g)] \cap [X \setminus Z(f)] \neq \emptyset$ . Therefore  $Z(fg) \neq X$ , i.e.,  $0 \neq fg \in E \cap I$ , and hence  $E$  is an essential ideal.  $\square$

By the above proposition, free ideals are essential ideals, and the ideals  $M_x$  and  $O_x$ , where  $x$  is a non-isolated point and hence non-maximal prime ideals, are also essential ideals.

The following proposition is proved in [4].

**Proposition 2.2.** *The socle  $C_F(X)$  of  $C(X)$  is a  $z$ -ideal, consisting of all functions that vanish everywhere except on a finite number of points of  $X$ .*

It is easy to see that a finite intersection of essential ideals in any commutative ring is an essential ideal. But even a countable intersection of essential ideals need not be an essential ideal. For example, the ideal  $O_r$  for any rational  $0 \leq r \leq 1$  is an essential ideal in  $C(\mathcal{R})$ . But  $I = \bigcap_{0 \leq r \leq 1} O_r$  is not an essential ideal, for  $\cap Z[I] = [0, 1]$  and  $\text{int}[0, 1] \neq \emptyset$ .

The following result is the consequence of Propositions 2.1 and 2.2.

**Corollary 2.3.** *Every intersection of essential ideals of  $C(X)$  is an essential ideal if and only if the set of isolated points of  $X$  is dense in  $X$ .*

The following theorem characterizes those compact spaces  $X$  for which every countable intersection of essential ideals of  $C(X)$  is an essential ideal.

**Theorem 2.4.** *Let  $X$  be a compact space. Then every countable intersection of essential ideals of  $C(X)$  is an essential ideal if and only if every first category subset of  $X$  is nowhere dense in  $X$ .*

*Proof.* First, suppose that every countable intersection of essential ideals in  $C(X)$  is an essential ideal, and let  $(A_n)_{n=1}^\infty$  be a sequence of nowhere dense subsets of  $X$ . By [2, Lemma 1.6],  $\cap Z[O_{A_n}] = \text{cl}(A_n)$ . Since  $\text{int}(\text{cl}[A_n]) = \emptyset$ , then  $O_{A_n}$  is an essential ideal by Proposition 2.1. By hypothesis,  $E = \bigcap_{n=1}^\infty O_{A_n}$  is an essential ideal. But again by [2, Lemma 1.6],  $\cap Z[E] = \text{cl}(\bigcup_{n=1}^\infty A_n)$  and by Proposition 2.1, we must have  $\text{int}[\text{cl}(\bigcup_{n=1}^\infty A_n)] = \emptyset$ , i.e.,  $\bigcup_{n=1}^\infty A_n$  is nowhere dense.

Conversely, let every first category subset of  $X$  be nowhere dense in  $X$ , and let  $(E_n)$  be a sequence of essential ideals in  $C(X)$ . Letting  $\cap Z[E_n] = A_n$ , then  $\text{int}(A_n) = \emptyset$ , and by the McKnight Theorem [2, Theorem 1.3],  $O_{A_n} \subseteq E_n \subseteq M_{A_n}$ , hence  $O_A \subseteq \bigcap_{n=1}^\infty E_n$ , where  $A = \bigcup_{n=1}^\infty A_n$ . Now we have  $\cap Z[O_A] = \text{cl}(A)$ , and since  $A$  is a first category set, then  $\text{int}(\text{cl}[A]) = \emptyset$ , i.e.,  $O_A$  is an essential ideal. This implies that  $\bigcap_{n=1}^\infty E_n$  is also an essential ideal.  $\square$

## 3. ALMOST LOCALLY COMPACT SPACES

**Definition 3.1.** A Hausdorff space  $X$  is said to be almost locally compact if every non-empty open set of  $X$  contains a non-empty open set with compact closure.

Our definition of almost locally compact spaces is equivalent to the one given in [6, p. 224]. Next, we give some examples of completely regular spaces which are almost locally compact but not locally compact.

**Example 3.1.** Clearly every locally compact space is almost locally compact, but not conversely.

- Let  $S$  be an uncountable space in which all points are isolated except for a distinguished point  $s$ , a neighborhood of  $s$  being any set containing  $s$  whose complement is countable.
- Make the real numbers into a topological space by taking as a base for open sets the family of all open intervals and  $\{\{r\} : r \in \mathcal{Q}\}$ .

**Example 3.2.** Clearly every open subspace of an almost locally compact space is an almost locally compact space, but in the preceding example, since  $\mathcal{R} \setminus \mathcal{Q}$  is not almost locally compact, we conclude that the closed subspaces of an almost locally compact space need not be almost locally compact.

The proof of the following proposition is trivial.

**Proposition 3.1.** (i) *The free union  $\bigcup_{s \in S} X_s$  is almost locally compact if and only if each  $X_s$  is.*

(ii) *The cartesian product  $\prod_{s \in S} X_s$ , where for every  $s \in S$ ,  $X_s \neq \emptyset$ , is almost locally compact if and only if all spaces  $X_s$  are almost locally compact and there exists a finite set  $S_0 \subseteq S$  such that  $X_s$  is compact for  $s \in S \setminus S_0$ .*

The next result is an algebraic characterization of almost locally compact spaces.

**Theorem 3.2.** *For every completely regular space  $X$ , the following statements are equivalent:*

- $X$  is an almost locally compact space.
- $C_K(X)$  is an essential ideal.
- $C_\infty(X)$  is an essential ideal.

*Proof.* (i)  $\Rightarrow$  (ii) Suppose that  $X$  is an almost locally compact space. We will prove that for every non-unit  $g \in C(X)$ ,  $C_K(X) \cap (g) \neq (0)$ . Since  $X \setminus Z(g)$  is an open set, then by regularity of  $X$ , there is an open set  $U$ , where  $U \subseteq \text{cl}(U) \subseteq X \setminus Z(g)$ , and there is an open set  $V$  such that  $\text{cl}(V)$  is compact and  $V \subseteq U$ . Then  $V \subseteq \text{cl}(V) \subseteq \text{cl}(U) \subseteq X \setminus Z(g)$ . Define  $f \in C(X)$  such that  $f(X \setminus V) = \{0\}$  and  $f(x) = 1$  for some  $x \in V$ . Since  $\text{cl}[X \setminus Z(f)] \subseteq \text{cl}(V)$ , and  $\text{cl}(V)$  is compact, so  $\text{cl}[X \setminus Z(f)]$  is also compact, i.e.,  $f \in C_K(X)$ . Hence  $fg \neq 0$  and  $fg \in C_K(X) \cap (g)$ .

(ii)  $\Rightarrow$  (iii) Since  $C_K(X) \subseteq C_\infty(X)$ , clearly  $C_\infty(X)$  must be an essential ideal.

(iii)  $\Rightarrow$  (i) Let  $U$  be a proper open set in  $X$ . By the regularity of  $X$ , there is a non-empty open set  $V$  such that  $V \subseteq \text{cl}(V) \subseteq U$ . Now find  $f \in C(X)$ , where  $f[\text{cl}(V)] = \{1\}$  and  $f(x) = 0$  for some  $x \notin U$ . If  $\text{cl}(V)$  is compact, there is nothing to be proved. Suppose  $\text{cl}(V)$  is not compact. If  $V \subseteq Z(h)$  for every  $h \in C_\infty(X)$ , then  $V \subseteq \cap Z[C_\infty(X)]$ , which implies that  $C_\infty(X)$  is not an essential ideal. Therefore there is some  $h \in C_\infty(X)$  such that  $V \cap [X \setminus Z(h)] \neq \emptyset$ , i.e., there is some  $x_0 \in V$  for which  $h(x_0) \neq 0$ . Clearly  $hf \in C_\infty(X)$ . So  $W = \{x : |h(x)f(x)| \geq \frac{1}{n_0}\}$  is

compact, where  $|h(x_0)| > \frac{1}{n_0}$ . If  $W' = \{x : |h(x)f(x)| > \frac{1}{n_0}\}$ , then  $W' \cap V$  is a non-empty open set in  $U$  and, since  $\text{cl}(W' \cap V) \subseteq W \cap \text{cl}(V)$  and  $W \cap \text{cl}(V)$  is a closed subset of the compact set  $W$ , then  $W \cap \text{cl}(V)$ , and consequently  $\text{cl}(W' \cap V)$  is compact, i.e.,  $X$  is an almost locally compact space.  $\square$

#### 4. PSEUDO-DISCRETE SPACES

**Definition 4.1.** A completely regular space  $X$  is said to be a pseudo-discrete space if every compact subset of  $X$  has finite interior. Clearly the class of pseudo-discrete spaces contains the class of  $P$ -spaces. Every pseudo-finite space (a space in which every compact subset is finite) is a pseudo-discrete space, but not conversely. For example, the space  $\mathcal{Q}$  of rational numbers is a pseudo-discrete space which is not pseudo-finite. For another example of a pseudo-discrete space, consider the free union  $D \dot{\cup} \mathcal{Q}$ , where  $D$  is a discrete space and  $\mathcal{Q}$  is the space of rational numbers.

**Proposition 4.1.** *Every locally compact pseudo-discrete space is discrete.*

*Proof.* Let  $X$  be a locally compact, pseudo-discrete space and let  $x \in X$ . Then  $x$  has a neighborhood  $U$  whose closure is compact, hence  $\text{int}(\text{cl}(U))$  is finite. But  $U \subseteq \text{int}(\text{cl}(U))$ , therefore  $U$  is finite, i.e.,  $x$  is an isolated point.  $\square$

The proof of the following proposition is easy.

**Proposition 4.2.** *Every open subspace of a pseudo-discrete space is a pseudo-discrete space.*

**Example 4.1.** Proposition 4.1 is not true for almost locally compact spaces, as the first example of 3.1 shows. In Proposition 4.2, if we consider a closed subspace instead of an open subspace, then it need not be a pseudo-discrete space. E.g., the closed subspace of  $\mathcal{Q}$  given by  $Y = \{\frac{1}{n} : n \in \mathcal{N}\} \cup \{0\}$  is not pseudo-discrete.

The following results show how to construct several pseudo-discrete spaces.

**Proposition 4.3.** *The free union  $\dot{\bigcup}_{i \in S} X_i$  is pseudo-discrete if and only if each  $X_i$  is.*

*Proof.* If  $\dot{\bigcup}_{i \in S} X_i$  is pseudo-discrete, then by Proposition 4.2, each  $X_i$  is also pseudo-discrete. Conversely, if each  $X_i$  is pseudo-discrete and  $A$  is a compact subset of  $\dot{\bigcup}_{i \in S} X_i$ , then the set  $\{i \in S : A \cap X_i \neq \emptyset\}$  is finite. Since  $\text{int}_{X_i}(A \cap X_i)$  is finite, then  $\text{int}(A) = \bigcup_{i \in S} \text{int}_{X_i}(A \cap X_i)$  is finite, i.e.,  $\dot{\bigcup}_{i \in S} X_i$  is pseudo-discrete.  $\square$

**Proposition 4.4.** (i) *If there is an infinite subset  $S_0$  of  $S$  such that for every  $i \in S_0$ ,  $X_i$  is non-compact, then  $\prod_{i \in S} X_i$  is pseudo-discrete.*

(ii) *If there exists  $i \in S$  such that every compact subset of  $X_i$  has empty interior, then  $\prod_{i \in S} X_i$  is pseudo-discrete.*

(iii)  $\prod_{i=1}^n X_i$  is pseudo-discrete if and only if each  $X_i$  is.

Karamzadeh and Rostami in [4] have shown that  $C_K(X) = C_F(X)$  for a large class of topological spaces, and have asked for a topological characterization of all these spaces. The next result settles this question.

**Theorem 4.5.** (i)  $C_K(X) = C_F(X)$  if and only if  $X$  is a pseudo-discrete space.

(ii)  $C_\infty(X) = C_F(X)$  if and only if  $X$  is a pseudo-discrete space with only a finite number of isolated points.

*Proof.* (i) Let  $X$  be a pseudo-discrete space and  $f \in C_K(X)$ . Then  $\text{cl}[X \setminus Z(f)]$  is compact and hence  $\text{int cl}[X \setminus Z(f)]$  is finite. But  $X \setminus Z(f) \subseteq \text{int cl}[X \setminus Z(f)]$  implies that  $X \setminus Z(f)$  is finite, i.e.,  $f \in C_F(X)$ . Therefore  $C_K(X) = C_F(X)$ .

Conversely, suppose that  $C_K(X) = C_F(X)$  and  $A$  is a compact subset of  $X$ . Suppose  $\text{int}(A)$  is not finite and assume that  $\{x_1, x_2, \dots, x_n, \dots\}$  is an infinite subset of  $\text{int}(A)$ . Now for each positive integer  $n$ , we define  $f_n \in C(X)$  such that  $f_n(x_n) = 1$  and  $f_n[X \setminus \text{int}(A)] = \{0\}$ . Then  $f = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{f_n^2}{1+f_n^2}$  is clearly continuous,  $f[X \setminus \text{int}(A)] = \{0\}$ , and for every positive integer  $n$ ,  $f(x_n) \neq 0$ . Since  $\text{cl}[X \setminus Z(f)] \subseteq \text{cl}[\text{int}(A)] \subseteq A$ , then  $\text{cl}[X \setminus Z(f)]$  is compact, i.e.,  $f \in C_K(X)$ . But  $C_K(X) = C_F(X)$ , therefore  $X \setminus Z(f)$  is finite, a contradiction.

(ii) Let  $C_{\infty}(X) = C_F(X)$ , then  $C_K(X) = C_F(X)$  and by part (i),  $X$  must be a pseudo-discrete space. Now suppose that the set of isolated points of  $X$  contains an infinite subset, say  $\{x_1, x_2, \dots, x_n, \dots\}$ . Define  $f_n \in C(X)$  to be such that  $f_n(x_n) = \frac{1}{n}$  and  $f_n[X \setminus \{x_n\}] = \{0\}$ . Let  $f = \sum_{n=1}^{\infty} f_n$ . Clearly  $f \in C(X)$ ,  $f(x_n) = \frac{1}{n}$ , and  $f[X \setminus \{x_1, x_2, \dots, x_n, \dots\}] = \{0\}$ . For every  $m$ , we have

$$\left\{x : |f(x)| \geq \frac{1}{m}\right\} = \{x_1, x_2, \dots, x_m\},$$

i.e., for each  $m$ ,  $\{x : |f(x)| \geq \frac{1}{m}\}$  is compact, and hence  $f \in C_{\infty}(X)$ . But

$$X \setminus Z(f) = \{x_1, x_2, \dots, x_n, \dots\}$$

which implies that  $f \notin C_F(X)$ , a contradiction.

Conversely, let  $X$  be a pseudo-discrete space with only a finite number of isolated points and  $f \in C_{\infty}(X)$ , but  $f \notin C_F(X)$ . Since  $A_n = \{x : |f(x)| > \frac{1}{n}\} \subseteq \{x : |f(x)| \geq \frac{1}{n}\}$  and  $\{x : |f(x)| \geq \frac{1}{n}\}$  is compact, then its interior is finite ( $X$  is a pseudo-discrete space). Hence  $A_n$  is a finite open set for every positive integer  $n$ . On the other hand,  $f \notin C_F(X)$ , i.e., there is an infinite set  $\{x_1, x_2, \dots, x_n, \dots\}$ , where for each  $n$ ,  $f(x_n) \neq 0$ . But for every  $n$ , there is an integer  $k_n$  such that  $|f(x_n)| > \frac{1}{k_n}$ , i.e.,  $x_n \in A_{k_n}$ . Since  $A_{k_n}$  is an open set, it follows that  $x_n$  is an isolated point, a contradiction.  $\square$

In [3, 7F.5], it is stated that  $C_K(\mathcal{Q}) = C_{\infty}(\mathcal{Q})$ . More generally, we have the following:

**Corollary 4.6.** (i)  $C_F(\mathcal{Q}) = C_K(\mathcal{Q}) = C_{\infty}(\mathcal{Q})$ .

(ii) If  $X$  is a pseudo-discrete space with only a finite number of isolated points, then  $C_F(X) = C_K(X) = C_{\infty}(X)$ .

*Proof.* By Theorem 4.5, this is evident.  $\square$

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