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HEREDITARY NOETHERIAN CATEGORIES WITH A TILTING COMPLEX

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In memory of Maurice Auslander

Abstract. We are characterizing the categories of coherent sheaves on a weighted projective line as the small hereditary noetherian categories without projectives and admitting a tilting complex. The paper is related to recent work with de la Peña (Math. Z., to appear) characterizing finite dimensional algebras with a sincere separating tubular family, and further gives a partial answer to a question of Happel, Reiten, Smalø (Mem. Amer. Math. Soc. **120** (1996), no. 575) regarding the characterization of hereditary categories with a tilting object.

A characterization of weighted projective lines

We are going to characterize the categories $\mathrm{coh}(\mathbb{X})$ of coherent sheaves on a weighted projective line $\mathbb{X}$ [3, 4].

Theorem 1. *Let k be an algebraically closed field. For a small connected abelian k-category $\mathcal{H}$ with finite dimensional morphism and extension spaces the following assertions are equivalent:*

(i) *$\mathcal{H}$ is equivalent to the category of coherent sheaves on a weighted projective line.*

(ii) *Each object of $\mathcal{H}$ is noetherian. $\mathcal{H}$ is hereditary, has no non-zero projectives, and admits a tilting complex.*

(iii) *Each object of $\mathcal{H}$ is noetherian, moreover*

  (a) *There exists an equivalence $\tau\colon \mathcal{H} \to \mathcal{H}$ (Auslander-Reiten translation) such that Serre duality $\mathrm{D}\,\mathrm{Ext}^1(A,B) \cong \mathrm{Hom}(B,\tau A)$ holds functorially in $A, B \in \mathcal{H}$.*

  (b) *The Grothendieck group $\mathrm{K}_0(\mathcal{H})$ is finitely generated free, and the Euler form $\langle -,-\rangle\colon \mathrm{K}_0(\mathcal{H}) \times \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$ given on classes of objects of $\mathcal{H}$ by $\langle [X],[Y]\rangle = \dim_k \mathrm{Hom}(X,Y) - \dim_k \mathrm{Ext}^1(X,Y)$ is non-degenerate of determinant ± 1.*

  (c) *$\mathcal{H}$ has an object without self-extensions which is not of finite length.*

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The result includes a characterization of the category $\mathrm{coh}(\mathbb{P}_1(k))$ of coherent sheaves on the projective line adding the requirement: *If F is not of finite length and S is simple in $\mathcal{H}$, then* $\mathrm{Hom}(F,S) \neq 0$. The theorem also gives a partial answer to a question raised by Happel, Reiten, Smalø [6] about the hereditary abelian k-categories admitting a tilting object.

For a finite dimensional k-algebra $\Sigma, \mathrm{mod}(\Sigma)$ denotes the category of finite dimensional right modules. For a k-vectorspace X we denote its k-dual $\mathrm{Hom}_k(X,k)$ by $\mathrm{D}X$. For an abelian category $\mathcal{H}$ its derived category of bounded complexes is denoted $D^b(\mathcal{H})$, and $X \mapsto X[n]$ denotes the nth iterate of the translation functor for $D^b(\mathcal{H})$. An object X in $\mathcal{H}$ is called *noetherian* if each ascending chain of subobjects of X is stationary. We say that $E \in \mathcal{H}$ is *without self-extensions* if $\mathrm{Ext}^n(E,E) = 0$ holds for each $n \neq 0$. A k-category $\mathcal{H}$ is *connected* if a representation of $\mathcal{H}$ as a coproduct $\mathcal{H} \cong \mathcal{A} \coprod \mathcal{B}$ implies $\mathcal{A} = 0$ or $\mathcal{B} = 0$. We say that $\mathcal{H}$ is *hereditary* if $\mathrm{Ext}^2_{\mathcal{H}}(-,-) = 0$.

A *tilting object* Σ in $\mathcal{H}$ is defined by the fact that the right derived functor of $\mathrm{Hom}(\Sigma,-)\colon \mathcal{H} \to \mathrm{mod}(\mathrm{End}(\Sigma))$ induces an equivalence of triangulated categories $D^b(\mathcal{H}) \to D^b(\mathrm{mod}(\mathrm{End}(\Sigma)))$. More generally, we say that $\mathcal{H}$ admits a *tilting complex* if for some finite dimensional algebra Σ there exists an equivalence $\Phi\colon D^b(\mathrm{mod}(\Sigma)) \to D^b(\mathcal{H})$ of triangulated categories. Here, Σ has an interpretation as the endomorphism ring of the tilting complex $\Phi(\Sigma)$ of $D^b(\mathcal{H})$. Whenever convenient, we prefer to view a tilting complex as a full subcategory of $D^b(\mathcal{H})$ with finitely many objects, thus avoiding the use of endomorphism rings.

This work has substantially profited from the collaboration with José Antonio de la Peña on finite dimensional algebras with a sincere separating tubular family [9]; we further thank Hagen Meltzer for critical comments on a first draft of this paper.

Proof of the characterization

Let $\mathcal{H}_0$ be the full subcategory consisting of all objects of $\mathcal{H}$ having finite length.

Lemma 1. *Assume that $\mathcal{H}$ satisfies condition* (iii)(a). *Then $\mathcal{H}_0$ is uniserial, i.e. each object $U \in \mathcal{H}_0$ has a unique finite composition series. Accordingly, for some index set $\mathbb{X}$ the category $\mathcal{H}_0$ decomposes into a coproduct $\mathcal{H}_0 = \coprod_{x\in\mathbb{X}} \mathcal{U}_x$ of connected uniserial categories $\mathcal{U}_x$, whose indecomposables form stable tubes.*

If moreover $\mathrm{K}_0(\mathcal{H})$ is finitely generated, then for each $x \in \mathbb{X}$ the simple objects in $\mathcal{U}_x$ form an Auslander-Reiten orbit of finite cardinality $u(x)$. Moreover $u(x) = 1$ for all but finitely many $x \in \mathbb{X}$.

Proof. In view of (iii)(a) the category $\mathcal{H}$ is hereditary. Hence $\mathcal{H}_0$ is a hereditary abelian category with almost-split sequences where each object has finite length, and the Auslander-Reiten translation $\tau\colon \mathcal{H}_0 \to \mathcal{H}_0$ is an equivalence. For simple objects S_i, S_j this implies that $\mathrm{Ext}^1(S_j,S_i) \neq 0$ if and only if $S_i \cong \tau S_j$, moreover that in this case the Ext-space is one-dimensional. Uniseriality of $\mathcal{H}_0$ now follows easily [2].

For each $x \in \mathbb{X}$ we fix a simple object S_x from $\mathcal{U}_x$, and collect a representative system (S_α), $\alpha \in I$, of the remaining simple objects. I admits a linear ordering such that $\langle [S_\alpha],[S_\beta]\rangle = 1$ for $\alpha = \beta$, but is zero for $\alpha > \beta$. This implies that the classes $[S_\alpha]$, $\alpha \in I$, are linearly independent in $\mathrm{K}_0(\mathcal{H})$, and proves the claim on u. $\square$

Proof of (ii) $\Rightarrow$ (iii). In view of $D^b(\mathcal{H}) \cong D^b(\mathrm{mod}(\Sigma))$ the group $\mathrm{K}_0(\mathcal{H})$ is isomorphic to $\mathrm{K}_0(\mathrm{mod}(\Sigma))$, which is finitely generated free. As a tilting complex for a hereditary category, the algebra Σ has finite global dimension which implies the assertion on the Euler form. Happel's theorem [5] shows that $D^b(\mathcal{H}) = D^b(\mathrm{mod}(\Sigma))$ has Auslander-Reiten triangles, and that further the Auslander-Reiten translation τ is an equivalence for $D^b(\mathcal{H})$. Next we prove that $\tau(\mathcal{H})$ is contained in $\mathcal{H}$. Let X be indecomposable in $\mathcal{H}$, we get $\tau X = Y[n]$ for some $Y \in \mathcal{H}$ and $n \in \mathbb{Z}$. Because

$$\mathrm{D}\,\mathrm{Ext}^{n+1}_{\mathcal{H}}(X, Y) \cong \mathrm{Hom}_{D^b(\mathcal{H})}(Y[n+1], \tau X[1]) \neq 0,$$

$n \geq 1$ is impossible for a hereditary category. Assuming $n \leq -1$ implies

$$\mathrm{D}\,\mathrm{Ext}^{1}_{\mathcal{H}}(X, Z) \cong \mathrm{Hom}_{D^b(\mathcal{H})}(Z, Y[n]) = 0$$

for each $Z \in \mathcal{H}$, so leads to a non-zero projective X in $\mathcal{H}$. This proves the assertion on Serre duality for $\mathcal{H}$, which in turn implies the existence of almost-split sequences in $\mathcal{H}$ (cf. [3]).

Finally assume that condition (iii)(c) is violated. Then $D^b(\mathcal{H})$ has a tilting complex lying in $D^b(\mathcal{H}_0)$, which involving Lemma 1 implies that $\mathcal{H} = \mathcal{H}_0$ is connected uniserial, hence the existence of a simple object S of τ-period p such that the classes $[S], [\tau S], \ldots, [\tau^{p-1} S]$ form a $\mathbb{Z}$-basis of $\mathrm{K}_0(\mathcal{H})$. Then $\mathbf{a} = \sum_{j=0}^{p-1} [\tau^j S]$ is a non-zero element of $\mathrm{K}_0(\mathcal{H})$ with $\langle \mathbf{a}, - \rangle = 0$, contradiction. □

The proof of implication (iii) $\Rightarrow$ (i) will occupy the rest of the paper. Each object X from $\mathcal{H}$ has a largest subobject tX of finite length (its *torsion part*), and X/tX has no simple submodules, i.e. is *torsionfree*. The full *subcategory of torsionfree objects* is denoted $\mathcal{H}_+$.

Lemma 2. *An indecomposable object X of $\mathcal{H}$ is either torsion ($X \in \mathcal{H}_0$) or torsionfree ($X \in \mathcal{H}_+$).*

Proof. Since $\mathrm{D}\,\mathrm{Ext}^1(X/tX, tX) = \mathrm{Hom}(\tau^-(tX), X/tX) = 0$ the sequence $0 \to tX \to X \to X/tX \to 0$ splits. □

Lemma 3. *There is a surjective linear mapping* $\mathrm{rk}\colon \mathrm{K}_0(\mathcal{H}) \to \mathbb{Z}$, $[X] \mapsto \mathrm{rk}\, X$, *called* ***rank function****, such that* $\mathrm{rk}\,\tau X = \mathrm{rk}\, X$ *for each* $X \in \mathcal{H}$, rk *vanishes on* $\mathcal{H}_0$, *and* $\mathrm{rk}\, F > 0$ *for each* $0 \neq F \in \mathcal{H}_+$.

Proof (de la Peña). For $x \in \mathbb{X}$ and $S_x \in \mathcal{U}_x$ simple, we define $w_x = \sum_{j=1}^{\mathcal{U}(x)} [\tau^j S_x]$ in $\mathrm{K}_0(\mathcal{H})$. Note that each w_x is stable under the mapping induced by τ on $\mathrm{K}_0(\mathcal{H})$. We choose $x_1, \ldots, x_s$ from $\mathbb{X}$ such that the subgroup of $\mathrm{K}_0(\mathcal{H})$ generated by $w_{x_1}, \ldots, w_{x_s}$ contains all w_x ($x \in \mathbb{X}$) and put $w = w_{x_1} + \cdots + w_{x_s}$.

By noetherianness each non-zero $F \in \mathcal{H}_+$ has a simple quotient S_x, hence $\langle [F], w_x \rangle > 0$, which implies $\langle [F], w_{x_i} \rangle > 0$ for some $i = 1, \ldots, s$, and finally $\langle [F], w \rangle > 0$. If ε denotes the index of $\langle \mathrm{K}_0(\mathcal{H}), w \rangle$ in $\mathbb{Z}$ we may take $\mathrm{rk}\,\mathbf{a} = 1/\varepsilon \cdot \langle \mathbf{a}, w \rangle$. □

Lemma 4. *Assume $B \in \mathcal{H}_+$ and $\mathrm{Ext}^1(\mathcal{U}_x, B) \neq 0$. For each $n \geq 0$ there exists an exact sequence $0 \to B \to B_n \to U_n \to 0$ with $B_n \in \mathcal{H}_+, U_n \in \mathcal{U}_x$ and $[U_n] = n \cdot w_x$.*

Proof. Let S_x be simple in $\mathcal{U}_x$ with $\mathrm{Ext}^1(S_x, B) \neq 0$. Choose non-split exact sequences $0 \to B'_i \to B'_{i+1} \to \tau^{-i} S_x \to 0$ ($i = 0, 1, 2, \ldots$), where $B'_0 = B$, and put $B_n = B'_{nu(x)}$. □

Lemma 5. *For each non-zero F in $\mathcal{H}_+$ and each $x \in \mathbb{X}$ there is a non-zero homomorphism from F into an object from $\mathcal{U}_x$. Accordingly, $[X] = 0$ in $\mathrm{K}_0(\mathcal{H})$ implies $X = 0$ in $\mathcal{H}$.*

Proof. We put

$$\begin{aligned}\mathbb{Y} &= \{y \in \mathbb{X} \mid \operatorname{Hom}(F, \mathcal{U}_y) = 0\},\\ \mathcal{A} &= \{A \in \mathcal{H} \mid \operatorname{Hom}(A, \mathcal{U}_y) = 0 \text{ for all } y \in \mathbb{Y}\},\\ \mathcal{B} &= \{B \in \mathcal{H} \mid \operatorname{Hom}(\mathcal{A}, B) = 0\}.\end{aligned}$$

Step 1. $\operatorname{Hom}(\mathcal{B}, \mathcal{U}_x) = 0$ for all $x \in \mathbb{X} \backslash \mathbb{Y}$.

Since $\tau \mathcal{U}_x = \mathcal{U}_x$ it is equivalent to prove that $\operatorname{Ext}^1(\mathcal{U}_x, \mathcal{B}) = 0$. Otherwise we obtain some B in $\mathcal{B}$ and exact sequences

$$0 \to B \to B_n \to U_n \to 0 \qquad (n = 1, 2, \dots),$$

having the properties of Lemma 4. We show that each B_n actually belongs to $\mathcal{B}$. Consider a morphism $f\colon A \to B_n$ with $A \in \mathcal{A}$, and from the pull-back

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & U_n \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & B_n \arrow[r] & U_n \arrow[r] & 0. \end{tikzcd}$$

Using $x \notin \mathbb{Y}$ we see that $A' \in \mathcal{A}$, hence $f' = 0$, so f induces a morphism $U_n \to B_n$ which is zero, implying $\operatorname{Hom}(\mathcal{A}, B_n) = 0$. Since $[B_n] = [B] + nw_x$, we obtain $\langle [F], B_n \rangle > 0$ for large n, thus contradicting $\operatorname{Hom}(\mathcal{A}, \mathcal{B}) = 0$.

Step 2. Each X in $\mathcal{H}$ has the form $X \cong A \oplus B$ with $A \in \mathcal{A}, B \in \mathcal{B}$.

By noetherianness X has a maximal subobject A from $\mathcal{A}$, accordingly X/A has no non-zero subobject from $\mathcal{A}$, hence belongs to $\mathcal{B}$. Obviously, $\mathcal{A}$ is closed under τ^-, therefore $\mathrm{D}\operatorname{Ext}^1(X/A, A) = \operatorname{Hom}(\tau^- A, X/A) = 0$, and $X \cong A \oplus X/A$.

Step 3. $\operatorname{Hom}(\mathcal{B}, \mathcal{A}) = 0$.

Otherwise we find $A \in \mathcal{A} \cap \mathcal{H}_+$, $B \in \mathcal{B} \cap \mathcal{H}_+$ and a non-zero morphism $f\colon B \to A$. Let $A' = \operatorname{im}(f)$. We choose $A' \subseteq A'' \subseteq A$ such that $A''/A' = t(A/A')$. Since $A \in \mathcal{A}$ and $A/A'' \in \mathcal{H}_+$ we see that $A'' \in \mathcal{A}$. Replacing A by A'' we may thus assume that $A/A' \in \mathcal{H}_0$, in fact $A/A' \in \coprod_{x \in \mathbb{X} \backslash \mathbb{Y}} \mathcal{U}_x$. By Step 1 the sequence $0 \to A' \to A \to A/A' \to 0$ splits, so A' belongs to $\mathcal{A}$. We thus arrive at an epimorphism $B \twoheadrightarrow A' \twoheadrightarrow S_x$ where S_x is simple in $\mathcal{U}_x$ and $x \in \mathbb{X} \backslash \mathbb{Y}$, contradiction, which proves the assertion of Step 3.

Combining Steps 2 and 3 with $\operatorname{Hom}(\mathcal{A}, \mathcal{B}) = 0$ we see that $\mathcal{H} = \mathcal{A} \coprod \mathcal{B}$. Since $\mathcal{H}$ is connected and $0 \neq F \in \mathcal{A}$, it follows that $\mathcal{B} = 0$, hence $\mathbb{Y} = \varnothing$ as claimed. $\square$

By $\mathrm{K}_0'(\mathcal{H})$ we denote the image of the natural map $\mathrm{K}_0(\mathcal{H}_0) \to \mathrm{K}_0(\mathcal{H})$, i.e. the subgroup of $\mathrm{K}_0(\mathcal{H})$ generated by the classes of simple objects.

Lemma 6. *There exists a torsionfree object L of rank one.* $\mathrm{K}_0(\mathcal{H}) = \mathbb{Z}[L] \oplus \mathrm{K}_0'(\mathcal{H})$. *Moreover for any $x, y \in \mathbb{X}, w_x$ and w_y are proportional.*

Proof. Choose a torsionfree object $L \neq 0$ of minimal rank. Each $\mathbf{a} \in \mathrm{K}_0'(\mathcal{H})$ satisfies $\operatorname{rk} \mathbf{a} = 0$, on the other hand $\operatorname{rk} L > 0$, which implies $\mathbb{Z}[L] \cap \mathrm{K}_0'(\mathcal{H}) = 0$. We show by induction on $r = \operatorname{rk} A$ that the class $[A]$ of an indecomposable object A belongs

to $\mathbb{Z}[L] + \mathrm{K}_0'(\mathcal{H})$. For $r = 0$ this follows from Lemma 3. For $r > 0$, A is torsionfree. We choose $x \in \mathbb{X}$ and a simple object S_x from $\mathcal{U}_x$ with $\mathrm{Ext}^1(S_x, A) \neq 0$. Invoking Lemma 4 we get an exact sequence $0 \to A \to \overline{A} \to U \to 0$, where $\overline{A}$ is torsionfree, U is in $\mathcal{U}_x$, and a non-zero homomorphism $f\colon L \to \overline{A}$. Because L has minimal rank and $\overline{A}$ is torsionfree, f is a monomorphism hence $C = \mathrm{coker}(f)$ has strictly smaller rank than A. By induction $[C]$, hence $[A]$, belongs to $\mathbb{Z}[L] \oplus \mathrm{K}_0'(\mathcal{H})$, proving the splitting of $\mathrm{K}_0(\mathcal{H})$. Surjectivity of the rank function now implies that L has rank one.

Since the linear forms $\langle -, w_x\rangle$ and $\langle -, w_y\rangle$ vanish on $\mathrm{K}_0'(\mathcal{H})$ they are determined by their non-zero value on $[L]$, hence are proportional. Because the Euler form on $\mathrm{K}_0(\mathcal{H})$ is non-degenerate, the corresponding assertion holds for w_x and w_y. □

Through the formation of perpendicular categories [4] we are going to reduce the weight function $u\colon \mathbb{X} \to \mathbb{N}$. An object E from $\mathcal{H}$ is called *exceptional* if E has a trivial endomorphism ring k and no self-extensions. Invoking Lemma 5 it follows from [7] that each indecomposable object $E \in \mathcal{H}$ without self-extensions is exceptional. Notice that every simple object S in $\mathcal{U}_x$ with $u(x) > 1$ is exceptional.

Proposition 1. *Let $x \in \mathbb{X}$ be exceptional, i.e. $u(x) > 1$, and let S be simple in $\mathcal{U}_x$.*

(a) *The right perpendicular category $\mathcal{H}' = S^\perp$ of S in $\mathcal{H}$, consisting of all $X \in \mathcal{H}$ with* $\mathrm{Hom}(S, X) = 0 = \mathrm{Ext}^1(S, X)$, *satisfies conditions* (iii) (a), (b), (c) *again.*

(b) *$\mathbb{X}$ serves naturally also as a parametrizing set for the decomposition of $\mathcal{H}_0' = S^\perp \cap \mathcal{H}_0$ into connected uniserial categories $\mathcal{U}_y' = S^\perp \cap \mathcal{U}_y$. The number of isomorphism classes of simples from $\mathcal{U}_y'$ is given by $u'(x) = u(x) - 1$ and $u'(y) = u(y)$ for $y \neq x$. Moreover, inclusion $\mathcal{H}' \hookrightarrow \mathcal{H}$ preserves the rank.*

(c) $\mathrm{K}_0(\mathcal{H}) = \mathbb{Z}[S] \oplus \mathrm{K}_0(\mathcal{H}')$.

Proof. Since $\mathcal{H}$ is hereditary and S is exceptional, inclusion $\mathcal{H}' \hookrightarrow \mathcal{H}$ admits a left adjoint ℓ and a right adjoint r [4]. Since S is simple, the category $\mathcal{H}_0'$ of torsion objects in $\mathcal{H}'$ agrees with $\mathcal{H}' \cap \mathcal{H}_0$, and therefore $\mathcal{H}_0' = \coprod_{y\in\mathbb{X}} \mathcal{U}_y'$ with $\mathcal{U}_y' = \mathcal{U}_y$ if $y \neq x$; further $\mathcal{U}_x' = S^\perp \cap \mathcal{U}_x$ is connected as well with $u(x) - 1$ isomorphic classes of simple objects. Moreover, $w_y' = w_y$ for each $y \in \mathbb{X}$, so inclusion $\mathcal{H}' \hookrightarrow \mathcal{H}$ preserves the rank.

If E is torsionfree in $\mathcal{H}$ without self-extensions, then $\mathrm{Hom}(S, E) = 0$ and ℓE is given by the middle term of the S-universal extension

$$0 \to E \to \ell E \to \mathrm{Ext}^1(S, E) \otimes_k S \to 0,$$

in particular ℓE is non-zero torsionfree. From the exactness of

$$0 = \mathrm{Ext}^1(E, E) \to \mathrm{Ext}^1(E, \ell E) \to \mathrm{Ext}^1(S, E) \otimes_k \mathrm{Ext}^1(E, S) = 0$$

we get $\mathrm{Ext}^1(E, \ell E) = 0$. By definition $\mathrm{Ext}^1(S, \ell E) = 0$ holds, hence $\mathrm{Ext}^1(\ell E, \ell E) = 0$, which proves the existence of a torsionfree object in $\mathcal{H}'$ without self-extensions.

Finally, assume that A and B are in $\mathcal{H}'$. Since

$$\mathrm{D}\,\mathrm{Ext}^1(A, B) = \mathrm{Hom}(\ell\tau_{\mathcal{H}}^- B, A) = \mathrm{Hom}(B, r\tau_{\mathcal{H}} A),$$

projectivity of A (resp. injectivity of B) in $\mathcal{H}'$ implies $r\tau_{\mathcal{H}} A = 0$ (resp. $\ell\tau_{\mathcal{H}}^- B = 0$) which, by the construction of r and ℓ [4], implies $A \in \mathcal{H}_0'$ (resp. $B \in \mathcal{H}_0'$). But any projective (resp. injective) in $\mathcal{U}_y'$, $y \in \mathbb{X}$, is zero, hence $A = 0$ (resp. $B = 0$). Invoking Auslander-Reiten theory this implies that $\tau_{\mathcal{H}'} = r\tau_{\mathcal{H}}$ and $\tau_{\mathcal{H}'}^- = \ell\tau_{\mathcal{H}}^-$ are inverse equivalences. For assertion (c) we refer to [4]. Further $\langle [S], \mathrm{K}_0(\mathcal{H}')\rangle = 0$ implies that the Euler forms on $\mathrm{K}_0(\mathcal{H})$ and $\mathrm{K}_0(\mathcal{H}')$ do have the same determinant. □

Repeating the process of Proposition 1 $\sum_{x\in\mathbb{X}}(u(x)-1)$ times leads to an exact abelian subcategory $\mathcal{H}'$ of $\mathcal{H}$ satisfying (iii) (a), (b), (c) with $u'(x)=1$ for each $x\in\mathbb{X}$.

Proposition 2. *Each torsionfree rank one object L of $\mathcal{H}'$ is exceptional. For each simple object S_x, $x\in\mathbb{X}$, we have $\dim_k \operatorname{Hom}(L,S_x)=1$ and $[S_x]=w$.*

Moreover there exist torsionfree rank one objects L_1,L_2 such that

(1)
$$\operatorname{Hom}(L_2,L_1)=0,\quad \dim_k\operatorname{Hom}(L_1,L_2)=2,\quad \operatorname{Ext}^1(L_i,L_j)=0\quad \textit{for } i,j=1,2.$$

Proof. By Lemma 6 the group $\mathrm{K}_0(\mathcal{H}')$ is free abelian of rank two. Fix a torsionfree object L of rank one, note that $\operatorname{End}(L)=k$ hence $\langle[L],[L]\rangle=1-g$, where $g=\dim_k\operatorname{Ext}^1(L,L)$, and define the *degree function* $\deg\colon \mathrm{K}_0(\mathcal{H}')\to\mathbb{Z}$ by

$$\deg z=\langle[L],z\rangle-(1-g)\operatorname{rk} z.$$

We thus obtain a Riemann-Roch formula

$$\langle\mathbf{a},\mathbf{b}\rangle=(1-g)\operatorname{rk}\mathbf{a}\operatorname{rk}\mathbf{b}+\begin{vmatrix}\operatorname{rk}\mathbf{a} & \operatorname{rk}\mathbf{b}\\ \deg\mathbf{a} & \deg\mathbf{b}\end{vmatrix}\quad \text{for all } \mathbf{a},\mathbf{b}\in\mathrm{K}_0(\mathcal{H}').$$

The existence of an exceptional object E of $\mathcal{H}'$ implies $0<\langle[E],[E]\rangle=(1-g)(\operatorname{rk}E)^2$ hence $g=0$, and so L is exceptional.

Next we choose $x\in\mathbb{X}$ such that $d=\deg w'_x$ is minimal. Note that $d>0$. For $y\in\mathbb{X}$ write $\deg w'_y=d\cdot q+r$ with $q,r\in\mathbb{N}$ and $0\le r<d$. Extending L by S_y and q times by S_x we obtain torsionfree rank one objects L',L'' with

$$[L']=[L]+w'_y,\qquad [L'']=[L]+q\cdot w'_x,$$

hence

$$\langle[L''],[L']\rangle=\langle[L],[L]\rangle+\langle[L],w'_y\rangle-q\cdot\langle[L],w_x\rangle\ge 1.$$

We thus arrive at an exact sequence $0\to L''\to L'\to U\to 0$, where $U\in\mathcal{H}'_0$ has degree $<d$. This shows $U=0$, hence $w'_y=q\cdot w'_x$. In view of Lemma 6 the classes $[L],[S_x]$ thus form a basis of $\mathrm{K}_0(\mathcal{H}')$, accordingly the determinant of the Euler form is $\pm1=\langle[L],[S_x]\rangle^2$, proving $\dim\operatorname{Hom}(L,S_x)=1$, therefore $\deg S_x=1$.

Put $L_1=L$, and let L_2 be the middle term of a non-split exact sequence $\mu\colon 0\to L_1\xrightarrow{\xi}L_2\to S_x\to 0$. Application of $\operatorname{Hom}(L_1,-)$ to μ shows $\operatorname{Hom}(L_1,L_2)\cong k^2$ and $\operatorname{Ext}^1(L_1,L_2)=0$, while application of $\operatorname{Hom}(-,L_1)$ leads to $\operatorname{Hom}(L_2,L_1)=0$ and $\operatorname{Ext}^1(L_2,L_1)=0$.

Extend ξ to a k-basis ξ,η of $\operatorname{Hom}(L_1,L_2)$. Invoking the properties of rank and degree, for each $a\in k$ the cokernel T_a of $\eta-a\xi\colon L_1\to L_2$ is seen to be simple. In fact, each simple object S_y ($y\neq x$) is isomorphic to some T_a: Setting $\xi^*=\operatorname{Hom}(\xi,S_y)$, $\eta^*=\operatorname{Hom}(\eta,S_y)$ we see that ξ^* is an isomorphism because $x\neq y$; moreover, since k is algebraically closed there exists an eigenvalue $a\in k$ such that the kernel $\operatorname{Hom}(T_a,S_y)$ of $\eta^*-a\xi^*\colon \operatorname{Hom}(L_2,S_y)\to\operatorname{Hom}(L_1,S_y)$ is non-zero. Hence $S_y\cong T_a$, in particular $[S_y]=[L_1]-[L_2]=[S_x]$. $\square$

Proposition 3. *$\mathcal{H}$ is a tilting object Δ*

$$\begin{tikzcd}
 & S_1^{[p_1-1]} \arrow[r, dash] & \cdots \arrow[r, dash] & S_1^{[1]} \\
L_1 \arrow[r, shift left, "\xi_1"] \arrow[r, shift right, "\xi_2"'] & L_2 \arrow[r, "\eta_2"] & S_2^{[p_2-1]} \arrow[r, dash] & \cdots \arrow[r, dash] & S_2^{[1]} \\
 & & \vdots & & \vdots \\
 & & S_t^{[p_t-1]} \arrow[r, dash] & \cdots \arrow[r, dash] & S_t^{[1]}
\end{tikzcd}$$

where L_1, L_2 are torsionfree rank one objects satisfying (1), and $S_i^{[l]}$ is an exceptional torsion object of length l.

Δ is a squid algebra in the sense of [10]*, i.e. equivalent to the path category of the above quiver subject to the relations*

$$\eta_1 \circ \xi_1 = 0, \quad \eta_2 \circ \xi_2 = 0, \quad \eta_i \circ (\xi_2 - \lambda_i \xi_1) = 0, \qquad i = 3, \dots, t, \tag{2}$$

where $\lambda_3, \dots, \lambda_t$ are pairwise distinct non-zero elements from k.

Δ also has a realization as a tilting object on the weighted projective line $\mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ given by the data $\mathbf{p} = (p_1, \dots, p_t)$, $\boldsymbol{\lambda} = (\lambda_3, \dots, \lambda_t)$.

Proof. Let $x_1, \dots, x_t$ denote the sequence of pairwise distinct exceptional elements from $\mathbb{X}$. We put $p_i = u(x_i)$, $\mathcal{U}_i = \mathcal{U}_{x_i}$. Let S_i be simple in $\mathcal{U}_i$ and $\mathcal{H}'$ denote the right perpendicular category with respect to the system

$$\tau^l S_i, \qquad l = 0, \dots, p_i - 2, \ i = 1, \dots, t, \tag{3}$$

and take a system L_1, L_2 of torsionfree rank one objects from $\mathcal{H}'$ as established in Proposition 2.

By induction on the rank it is easily verified (compare proof of Lemma 6) that each object from $\mathcal{H}$ is contained in the smallest subcategory of $\mathcal{H}$ which is closed under extensions, kernels of epimorphisms, cokernels of monomorphisms and contains

$$L_1, L_2, S_i, \tau S_i, \dots, \tau^{p_i - 2} S_i, \qquad i = 1, \dots, t; \tag{4}$$

in particular (4) generates $D^b(\mathcal{H})$.

Let $S_i^{[n]}$ denote the indecomposable object of $\mathcal{U}_i$ with top S_i and length n. Then the system

$$L_1, L_2, S_i = S_i^{[1]}, S_i^{[2]}, \dots, S_i^{[p_i - 1]}, \qquad i = 1, \dots, t, \tag{5}$$

generates $D^b(\mathcal{H})$ as well, moreover satisfies

$$\operatorname{Ext}^1(S_i^{[a]}, S_{i'}^{[a']}) = 0, \quad \operatorname{Ext}^1(S_i^{[a]}, L_j) = 0, \quad \operatorname{Ext}^1(L_j, S_i^{[a]}) = 0 \tag{6}$$

for all $i, i' = 1, \dots, t$, $a = 1, \dots, p_i - 1$, $a' = 1, \dots, p_{i'} - 1$, $j = 1, 2$.

Invoking $w_x = w$ for each $x \in \mathbb{X}$, we obtain exact sequences

$$0 \to L_1 \xrightarrow{\xi_i} L_2 \xrightarrow{\overline{\eta}_i} S_i^{[p_i]} \to 0 \qquad (1 \le i \le t),$$

defining t mutually disjoint 1-dimensional subspaces $k\xi_i$ of $\mathrm{Hom}(L_1, L_2)$, thus $\xi_i = \xi_2 - \lambda_i \xi_1$ for $2 \le i \le t$, where $0 = \lambda_2, \lambda_3, \ldots, \lambda_t$ are pairwise distinct. Setting $\eta_i = [L_2 \xrightarrow{\overline{\eta}_i} S_i^{[p_i]} \to S_i^{[p_i-1]}]$ and calculating dimensions of Hom-spaces by means of the Euler form further establishes that (2) is a complete set of relations for the subcategory Δ.

It is known (cf. [8]) and not difficult to prove that Δ can be realized as a tilting sheaf on the weighted projective line $\mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ arising from the above data $\mathbf{p} = (p_1, \ldots, p_t)$ and $\boldsymbol{\lambda} = (\lambda_3, \ldots, \lambda_t)$ [3]. Since moreover Δ has finite global dimension (actually $\mathrm{gl.dim}\ \Delta \le 2$) we may invoke Beilinson's lemma [1] to conclude that Δ is also a tilting object in $\mathcal{H}$. □

Proof of (iii) $\Rightarrow$ (i). Abbreviating $\mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ to $\mathbb{X}$ from now on, the right derived functor of $\mathrm{Hom}_{\mathbb{X}}(\Delta, -)\colon \mathrm{coh}(\mathbb{X}) \to \mathrm{mod}(\Delta)$ induces an equivalence of triangulated categories $\alpha\colon D^b(\mathrm{coh}(\mathbb{X})) \to D^b(\mathrm{mod}(\Delta))$, similarly the right derived functor of $\mathrm{Hom}_{\mathcal{H}}(\Delta, -)$ induces an equivalence $\beta\colon D^b(\mathcal{H}) \to D^b(\mathrm{mod}(\Delta))$. Since α and β preserve Δ the composition $\phi = \alpha^{-1} \circ \beta\colon D^b(\mathcal{H}) \to D^b(\mathrm{coh}(\mathbb{X}))$ preserves the rank and maps $\mathcal{H}_0$ onto the category $\mathrm{coh}_0(\mathbb{X})$ of finite length sheaves.

An indecomposable object X from $D^b(\mathcal{H})$ lies in $\mathcal{H}$ if and only if X is either in $\mathcal{H}_0$ or has rank > 0 and satisfies $\mathrm{Hom}(X, \mathcal{H}_0) \neq 0$. Similarly an indecomposable object Y from $D^b(\mathrm{coh}(\mathbb{X}))$ belongs to $\mathrm{coh}(\mathbb{X})$ if and only if either Y is in $\mathrm{coh}_0(\mathbb{X})$ or Y has rank > 0 and satisfies $\mathrm{Hom}(Y, \mathrm{coh}_0(\mathbb{X})) \neq 0$. Because $\phi(\mathcal{H}_0) = \mathrm{coh}_0(\mathbb{X})$, and ϕ preserves the rank, it hence follows that ϕ induces an equivalence $\mathcal{H} \to \mathrm{coh}(\mathbb{X})$, thus concluding the proof of the theorem. □

Given $\mathcal{H}$ satisfying (ii) or (iii), the theorem implies the existence of a "natural" bijection between the set $\mathbb{X}$, parametrizing the decomposition of $\mathcal{H}_0$ into connected components, and the projective line $\mathbb{P}_1(k)$, the point set underlying $\mathbb{X}$. To determine the weighted projective line corresponding to $\mathcal{H}$, one fixes a simple object S with $\tau S \cong S$ further a rank one object L of $\mathcal{H}$, and forms a non-split extension $0 \to L \to \overline{L} \to S \to 0$. The k-space $\mathrm{Hom}(L, \overline{L})$ has dimension two; each non-zero $v\colon L \to \overline{L}$ is a monomorphism whose cokernel is indecomposable of finite length, therefore belongs to $\mathcal{U}_{\psi(v)}$ for a unique $\psi(v) \in \mathbb{X}$. The induced mapping $\mathbb{P}_1(k) = \mathbb{P}(\mathrm{Hom}(L, \overline{L})) \to \mathbb{X}$, $kv \mapsto \psi(v)$, is a bijection turning the function u from Lemma 1 into a weight function on $\mathbb{P}_1(k)$, determining thus completely the data for the weighted projective line.

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