

## HEREDITARY NOETHERIAN CATEGORIES WITH A TILTING COMPLEX

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*In memory of Maurice Auslander*

ABSTRACT. We are characterizing the categories of coherent sheaves on a weighted projective line as the small hereditary noetherian categories without projectives and admitting a tilting complex. The paper is related to recent work with de la Peña (Math. Z., to appear) characterizing finite dimensional algebras with a sincere separating tubular family, and further gives a partial answer to a question of Happel, Reiten, Smalø (Mem. Amer. Math. Soc. **120** (1996), no. 575) regarding the characterization of hereditary categories with a tilting object.

### A CHARACTERIZATION OF WEIGHTED PROJECTIVE LINES

We are going to characterize the categories  $\text{coh}(\mathbb{X})$  of coherent sheaves on a weighted projective line  $\mathbb{X}$  [3, 4].

**Theorem 1.** *Let  $k$  be an algebraically closed field. For a small connected abelian  $k$ -category  $\mathcal{H}$  with finite dimensional morphism and extension spaces the following assertions are equivalent:*

- (i)  $\mathcal{H}$  is equivalent to the category of coherent sheaves on a weighted projective line.
- (ii) Each object of  $\mathcal{H}$  is noetherian.  $\mathcal{H}$  is hereditary, has no non-zero projectives, and admits a tilting complex.
- (iii) Each object of  $\mathcal{H}$  is noetherian, moreover
  - (a) There exists an equivalence  $\tau: \mathcal{H} \rightarrow \mathcal{H}$  (Auslander-Reiten translation) such that Serre duality  $D \text{Ext}^1(A, B) \cong \text{Hom}(B, \tau A)$  holds functorially in  $A, B \in \mathcal{H}$ .
  - (b) The Grothendieck group  $K_0(\mathcal{H})$  is finitely generated free, and the Euler form  $\langle -, - \rangle: K_0(\mathcal{H}) \times K_0(\mathcal{H}) \rightarrow \mathbb{Z}$  given on classes of objects of  $\mathcal{H}$  by  $\langle [X], [Y] \rangle = \dim_k \text{Hom}(X, Y) - \dim_k \text{Ext}^1(X, Y)$  is non-degenerate of determinant  $\pm 1$ .
  - (c)  $\mathcal{H}$  has an object without self-extensions which is not of finite length.

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The result includes a characterization of the category  $\text{coh}(\mathbb{P}_1(k))$  of coherent sheaves on the projective line adding the requirement: *If  $F$  is not of finite length and  $S$  is simple in  $\mathcal{H}$ , then  $\text{Hom}(F, S) \neq 0$ .* The theorem also gives a partial answer to a question raised by Happel, Reiten, Smalø [6] about the hereditary abelian  $k$ -categories admitting a tilting object.

For a finite dimensional  $k$ -algebra  $\Sigma$ ,  $\text{mod}(\Sigma)$  denotes the category of finite dimensional right modules. For a  $k$ -vector space  $X$  we denote its  $k$ -dual  $\text{Hom}_k(X, k)$  by  $DX$ . For an abelian category  $\mathcal{H}$  its derived category of bounded complexes is denoted  $D^b(\mathcal{H})$ , and  $X \mapsto X[n]$  denotes the  $n$ th iterate of the translation functor for  $D^b(\mathcal{H})$ . An object  $X$  in  $\mathcal{H}$  is called *noetherian* if each ascending chain of subobjects of  $X$  is stationary. We say that  $E \in \mathcal{H}$  is *without self-extensions* if  $\text{Ext}^n(E, E) = 0$  holds for each  $n \neq 0$ . A  $k$ -category  $\mathcal{H}$  is *connected* if a representation of  $\mathcal{H}$  as a coproduct  $\mathcal{H} \cong \mathcal{A} \coprod \mathcal{B}$  implies  $\mathcal{A} = 0$  or  $\mathcal{B} = 0$ . We say that  $\mathcal{H}$  is *hereditary* if  $\text{Ext}_{\mathcal{H}}^2(-, -) = 0$ .

A *tilting object*  $\Sigma$  in  $\mathcal{H}$  is defined by the fact that the right derived functor of  $\text{Hom}(\Sigma, -): \mathcal{H} \rightarrow \text{mod}(\text{End}(\Sigma))$  induces an equivalence of triangulated categories  $D^b(\mathcal{H}) \rightarrow D^b(\text{mod}(\text{End}(\Sigma)))$ . More generally, we say that  $\mathcal{H}$  admits a *tilting complex* if for some finite dimensional algebra  $\Sigma$  there exists an equivalence  $\Phi: D^b(\text{mod}(\Sigma)) \rightarrow D^b(\mathcal{H})$  of triangulated categories. Here,  $\Sigma$  has an interpretation as the endomorphism ring of the tilting complex  $\Phi(\Sigma)$  of  $D^b(\mathcal{H})$ . Whenever convenient, we prefer to view a tilting complex as a full subcategory of  $D^b(\mathcal{H})$  with finitely many objects, thus avoiding the use of endomorphism rings.

This work has substantially profited from the collaboration with José Antonio de la Peña on finite dimensional algebras with a sincere separating tubular family [9]; we further thank Hagen Meltzer for critical comments on a first draft of this paper.

#### PROOF OF THE CHARACTERIZATION

Let  $\mathcal{H}_0$  be the full subcategory consisting of all objects of  $\mathcal{H}$  having finite length.

**Lemma 1.** *Assume that  $\mathcal{H}$  satisfies condition (iii)(a). Then  $\mathcal{H}_0$  is uniserial, i.e. each object  $U \in \mathcal{H}_0$  has a unique finite composition series. Accordingly, for some index set  $\mathbb{X}$  the category  $\mathcal{H}_0$  decomposes into a coproduct  $\mathcal{H}_0 = \coprod_{x \in \mathbb{X}} \mathcal{U}_x$  of connected uniserial categories  $\mathcal{U}_x$ , whose indecomposables form stable tubes.*

*If moreover  $\text{K}_0(\mathcal{H})$  is finitely generated, then for each  $x \in \mathbb{X}$  the simple objects in  $\mathcal{U}_x$  form an Auslander-Reiten orbit of finite cardinality  $u(x)$ . Moreover  $u(x) = 1$  for all but finitely many  $x \in \mathbb{X}$ .*

*Proof.* In view of (iii)(a) the category  $\mathcal{H}$  is hereditary. Hence  $\mathcal{H}_0$  is a hereditary abelian category with almost-split sequences where each object has finite length, and the Auslander-Reiten translation  $\tau: \mathcal{H}_0 \rightarrow \mathcal{H}_0$  is an equivalence. For simple objects  $S_i, S_j$  this implies that  $\text{Ext}^1(S_j, S_i) \neq 0$  if and only if  $S_i \cong \tau S_j$ , moreover that in this case the Ext-space is one-dimensional. Uniseriality of  $\mathcal{H}_0$  now follows easily [2].

For each  $x \in \mathbb{X}$  we fix a simple object  $S_x$  from  $\mathcal{U}_x$ , and collect a representative system  $(S_\alpha)$ ,  $\alpha \in I$ , of the remaining simple objects.  $I$  admits a linear ordering such that  $\langle [S_\alpha], [S_\beta] \rangle = 1$  for  $\alpha = \beta$ , but is zero for  $\alpha > \beta$ . This implies that the classes  $[S_\alpha]$ ,  $\alpha \in I$ , are linearly independent in  $\text{K}_0(\mathcal{H})$ , and proves the claim on  $u$ .  $\square$

*Proof of (ii) ⇒ (iii).* In view of  $D^b(\mathcal{H}) \cong D^b(\text{mod}(\Sigma))$  the group  $K_0(\mathcal{H})$  is isomorphic to  $K_0(\text{mod}(\Sigma))$ , which is finitely generated free. As a tilting complex for a hereditary category, the algebra  $\Sigma$  has finite global dimension which implies the assertion on the Euler form. Happel’s theorem [5] shows that  $D^b(\mathcal{H}) = D^b(\text{mod}(\Sigma))$  has Auslander-Reiten triangles, and that further the Auslander-Reiten translation  $\tau$  is an equivalence for  $D^b(\mathcal{H})$ . Next we prove that  $\tau(\mathcal{H})$  is contained in  $\mathcal{H}$ . Let  $X$  be indecomposable in  $\mathcal{H}$ , we get  $\tau X = Y[n]$  for some  $Y \in \mathcal{H}$  and  $n \in \mathbb{Z}$ . Because

$$\text{DExt}_{\mathcal{H}}^{n+1}(X, Y) \cong \text{Hom}_{D^b(\mathcal{H})}(Y[n+1], \tau X[1]) \neq 0,$$

$n \geq 1$  is impossible for a hereditary category. Assuming  $n \leq -1$  implies

$$\text{DExt}_{\mathcal{H}}^1(X, Z) \cong \text{Hom}_{D^b(\mathcal{H})}(Z, Y[n]) = 0$$

for each  $Z \in \mathcal{H}$ , so leads to a non-zero projective  $X$  in  $\mathcal{H}$ . This proves the assertion on Serre duality for  $\mathcal{H}$ , which in turn implies the existence of almost-split sequences in  $\mathcal{H}$  (cf. [3]).

Finally assume that condition (iii)(c) is violated. Then  $D^b(\mathcal{H})$  has a tilting complex lying in  $D^b(\mathcal{H}_0)$ , which involving Lemma 1 implies that  $\mathcal{H} = \mathcal{H}_0$  is connected uniserial, hence the existence of a simple object  $S$  of  $\tau$ -period  $p$  such that the classes  $[S], [\tau S], \dots, [\tau^{p-1} S]$  form a  $\mathbb{Z}$ -basis of  $K_0(\mathcal{H})$ . Then  $\mathbf{a} = \sum_{j=0}^{p-1} [\tau^j S]$  is a non-zero element of  $K_0(\mathcal{H})$  with  $\langle \mathbf{a}, - \rangle = 0$ , contradiction.  $\square$

The proof of implication (iii) ⇒ (i) will occupy the rest of the paper. Each object  $X$  from  $\mathcal{H}$  has a largest subobject  $tX$  of finite length (its *torsion part*), and  $X/tX$  has no simple submodules, i.e. is *torsionfree*. The full subcategory of torsionfree objects is denoted  $\mathcal{H}_+$ .

**Lemma 2.** *An indecomposable object  $X$  of  $\mathcal{H}$  is either torsion ( $X \in \mathcal{H}_0$ ) or torsionfree ( $X \in \mathcal{H}_+$ ).*

*Proof.* Since  $\text{DExt}^1(X/tX, tX) = \text{Hom}(\tau^-(tX), X/tX) = 0$  the sequence  $0 \rightarrow tX \rightarrow X \rightarrow X/tX \rightarrow 0$  splits.  $\square$

**Lemma 3.** *There is a surjective linear mapping  $\text{rk}: K_0(\mathcal{H}) \rightarrow \mathbb{Z}$ ,  $[X] \mapsto \text{rk } X$ , called **rank function**, such that  $\text{rk } \tau X = \text{rk } X$  for each  $X \in \mathcal{H}$ ,  $\text{rk}$  vanishes on  $\mathcal{H}_0$ , and  $\text{rk } F > 0$  for each  $0 \neq F \in \mathcal{H}_+$ .*

*Proof* (de la Peña). For  $x \in \mathbb{X}$  and  $S_x \in \mathcal{U}_x$  simple, we define  $w_x = \sum_{j=1}^{\mathcal{U}(x)} [\tau^j S_x]$  in  $K_0(\mathcal{H})$ . Note that each  $w_x$  is stable under the mapping induced by  $\tau$  on  $K_0(\mathcal{H})$ . We choose  $x_1, \dots, x_s$  from  $\mathbb{X}$  such that the subgroup of  $K_0(\mathcal{H})$  generated by  $w_{x_1}, \dots, w_{x_s}$  contains all  $w_x$  ( $x \in \mathbb{X}$ ) and put  $w = w_{x_1} + \dots + w_{x_s}$ .

By noetherianness each non-zero  $F \in \mathcal{H}_+$  has a simple quotient  $S_x$ , hence  $\langle [F], w_x \rangle > 0$ , which implies  $\langle [F], w_{x_i} \rangle > 0$  for some  $i = 1, \dots, s$ , and finally  $\langle [F], w \rangle > 0$ . If  $\varepsilon$  denotes the index of  $\langle K_0(\mathcal{H}), w \rangle$  in  $\mathbb{Z}$  we may take  $\text{rk } \mathbf{a} = 1/\varepsilon \cdot \langle \mathbf{a}, w \rangle$ .  $\square$

**Lemma 4.** *Assume  $B \in \mathcal{H}_+$  and  $\text{Ext}^1(\mathcal{U}_x, B) \neq 0$ . For each  $n \geq 0$  there exists an exact sequence  $0 \rightarrow B \rightarrow B_n \rightarrow U_n \rightarrow 0$  with  $B_n \in \mathcal{H}_+, U_n \in \mathcal{U}_x$  and  $[U_n] = n \cdot w_x$ .*

*Proof.* Let  $S_x$  be simple in  $\mathcal{U}_x$  with  $\text{Ext}^1(S_x, B) \neq 0$ . Choose non-split exact sequences  $0 \rightarrow B'_i \rightarrow B'_{i+1} \rightarrow \tau^{-i} S_x \rightarrow 0$  ( $i = 0, 1, 2, \dots$ ), where  $B'_0 = B$ , and put  $B_n = B'_{nu(x)}$ .  $\square$

**Lemma 5.** *For each non-zero  $F$  in  $\mathcal{H}_+$  and each  $x \in \mathbb{X}$  there is a non-zero homomorphism from  $F$  into an object from  $\mathcal{U}_x$ . Accordingly,  $[X] = 0$  in  $K_0(\mathcal{H})$  implies  $X = 0$  in  $\mathcal{H}$ .*

*Proof.* We put

$$\begin{aligned} \mathbb{Y} &= \{y \in \mathbb{X} \mid \text{Hom}(F, \mathcal{U}_y) = 0\}, \\ \mathcal{A} &= \{A \in \mathcal{H} \mid \text{Hom}(A, \mathcal{U}_y) = 0 \text{ for all } y \in \mathbb{Y}\}, \\ \mathcal{B} &= \{B \in \mathcal{H} \mid \text{Hom}(\mathcal{A}, B) = 0\}. \end{aligned}$$

*Step 1.*  $\text{Hom}(\mathcal{B}, \mathcal{U}_x) = 0$  for all  $x \in \mathbb{X} \setminus \mathbb{Y}$ .

Since  $\tau\mathcal{U}_x = \mathcal{U}_x$  it is equivalent to prove that  $\text{Ext}^1(\mathcal{U}_x, \mathcal{B}) = 0$ . Otherwise we obtain some  $B$  in  $\mathcal{B}$  and exact sequences

$$0 \rightarrow B \rightarrow B_n \rightarrow U_n \rightarrow 0 \quad (n = 1, 2, \dots),$$

having the properties of Lemma 4. We show that each  $B_n$  actually belongs to  $\mathcal{B}$ . Consider a morphism  $f: A \rightarrow B_n$  with  $A \in \mathcal{A}$ , and from the pull-back

$$\begin{array}{ccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & U_n \longrightarrow 0 \\ & & \downarrow f' & & \downarrow f & & \parallel \\ 0 & \longrightarrow & B & \longrightarrow & B_n & \longrightarrow & U_n \longrightarrow 0. \end{array}$$

Using  $x \notin \mathbb{Y}$  we see that  $A' \in \mathcal{A}$ , hence  $f' = 0$ , so  $f$  induces a morphism  $U_n \rightarrow B_n$  which is zero, implying  $\text{Hom}(\mathcal{A}, B_n) = 0$ . Since  $[B_n] = [B] + nw_x$ , we obtain  $\langle [F], B_n \rangle > 0$  for large  $n$ , thus contradicting  $\text{Hom}(\mathcal{A}, \mathcal{B}) = 0$ .

*Step 2.* Each  $X$  in  $\mathcal{H}$  has the form  $X \cong A \oplus B$  with  $A \in \mathcal{A}, B \in \mathcal{B}$ .

By noetherianness  $X$  has a maximal subobject  $A$  from  $\mathcal{A}$ , accordingly  $X/A$  has no non-zero subobject from  $\mathcal{A}$ , hence belongs to  $\mathcal{B}$ . Obviously,  $\mathcal{A}$  is closed under  $\tau^-$ , therefore  $D \text{Ext}^1(X/A, A) = \text{Hom}(\tau^- A, X/A) = 0$ , and  $X \cong A \oplus X/A$ .

*Step 3.*  $\text{Hom}(\mathcal{B}, \mathcal{A}) = 0$ .

Otherwise we find  $A \in \mathcal{A} \cap \mathcal{H}_+, B \in \mathcal{B} \cap \mathcal{H}_+$  and a non-zero morphism  $f: B \rightarrow A$ . Let  $A' = \text{im}(f)$ . We choose  $A' \subseteq A'' \subseteq A$  such that  $A''/A' = t(A/A')$ . Since  $A \in \mathcal{A}$  and  $A/A' \in \mathcal{H}_+$  we see that  $A'' \in \mathcal{A}$ . Replacing  $A$  by  $A''$  we may thus assume that  $A/A' \in \mathcal{H}_0$ , in fact  $A/A' \in \coprod_{x \in \mathbb{X} \setminus \mathbb{Y}} \mathcal{U}_x$ . By Step 1 the sequence  $0 \rightarrow A' \rightarrow A \rightarrow A/A' \rightarrow 0$  splits, so  $A'$  belongs to  $\mathcal{A}$ . We thus arrive at an epimorphism  $B \rightarrow A' \rightarrow S_x$  where  $S_x$  is simple in  $\mathcal{U}_x$  and  $x \in \mathbb{X} \setminus \mathbb{Y}$ , contradiction, which proves the assertion of Step 3.

Combining Steps 2 and 3 with  $\text{Hom}(\mathcal{A}, \mathcal{B}) = 0$  we see that  $\mathcal{H} = \mathcal{A} \coprod \mathcal{B}$ . Since  $\mathcal{H}$  is connected and  $0 \neq F \in \mathcal{A}$ , it follows that  $\mathcal{B} = 0$ , hence  $\mathbb{Y} = \emptyset$  as claimed.  $\square$

By  $K'_0(\mathcal{H})$  we denote the image of the natural map  $K_0(\mathcal{H}_0) \rightarrow K_0(\mathcal{H})$ , i.e. the subgroup of  $K_0(\mathcal{H})$  generated by the classes of simple objects.

**Lemma 6.** *There exists a torsionfree object  $L$  of rank one.  $K_0(\mathcal{H}) = \mathbb{Z}[L] \oplus K'_0(\mathcal{H})$ . Moreover for any  $x, y \in \mathbb{X}, w_x$  and  $w_y$  are proportional.*

*Proof.* Choose a torsionfree object  $L \neq 0$  of minimal rank. Each  $\mathbf{a} \in K'_0(\mathcal{H})$  satisfies  $\text{rk } \mathbf{a} = 0$ , on the other hand  $\text{rk } L > 0$ , which implies  $\mathbb{Z}[L] \cap K'_0(\mathcal{H}) = 0$ . We show by induction on  $r = \text{rk } A$  that the class  $[A]$  of an indecomposable object  $A$  belongs

to  $\mathbb{Z}[L] + K'_0(\mathcal{H})$ . For  $r = 0$  this follows from Lemma 3. For  $r > 0$ ,  $A$  is torsionfree. We choose  $x \in \mathbb{X}$  and a simple object  $S_x$  from  $\mathcal{U}_x$  with  $\text{Ext}^1(S_x, A) \neq 0$ . Invoking Lemma 4 we get an exact sequence  $0 \rightarrow A \rightarrow \overline{A} \rightarrow U \rightarrow 0$ , where  $\overline{A}$  is torsionfree,  $U$  is in  $\mathcal{U}_x$ , and a non-zero homomorphism  $f: L \rightarrow \overline{A}$ . Because  $L$  has minimal rank and  $\overline{A}$  is torsionfree,  $f$  is a monomorphism hence  $C = \text{coker}(f)$  has strictly smaller rank than  $A$ . By induction  $[C]$ , hence  $[A]$ , belongs to  $\mathbb{Z}[L] \oplus K'_0(\mathcal{H})$ , proving the splitting of  $K_0(\mathcal{H})$ . Surjectivity of the rank function now implies that  $L$  has rank one.

Since the linear forms  $\langle -, w_x \rangle$  and  $\langle -, w_y \rangle$  vanish on  $K'_0(\mathcal{H})$  they are determined by their non-zero value on  $[L]$ , hence are proportional. Because the Euler form on  $K_0(\mathcal{H})$  is non-degenerate, the corresponding assertion holds for  $w_x$  and  $w_y$ .  $\square$

Through the formation of perpendicular categories [4] we are going to reduce the weight function  $u: \mathbb{X} \rightarrow \mathbb{N}$ . An object  $E$  from  $\mathcal{H}$  is called *exceptional* if  $E$  has a trivial endomorphism ring  $k$  and no self-extensions. Invoking Lemma 5 it follows from [7] that each indecomposable object  $E \in \mathcal{H}$  without self-extensions is exceptional. Notice that every simple object  $S$  in  $\mathcal{U}_x$  with  $u(x) > 1$  is exceptional.

**Proposition 1.** *Let  $x \in \mathbb{X}$  be exceptional, i.e.  $u(x) > 1$ , and let  $S$  be simple in  $\mathcal{U}_x$ .*

(a) *The right perpendicular category  $\mathcal{H}' = S^\perp$  of  $S$  in  $\mathcal{H}$ , consisting of all  $X \in \mathcal{H}$  with  $\text{Hom}(S, X) = 0 = \text{Ext}^1(S, X)$ , satisfies conditions (iii) (a), (b), (c) again.*

(b)  *$\mathbb{X}$  serves naturally also as a parametrizing set for the decomposition of  $\mathcal{H}'_0 = S^\perp \cap \mathcal{H}_0$  into connected uniserial categories  $\mathcal{U}'_y = S^\perp \cap \mathcal{U}_y$ . The number of isomorphism classes of simples from  $\mathcal{U}'_y$  is given by  $u'(x) = u(x) - 1$  and  $u'(y) = u(y)$  for  $y \neq x$ . Moreover, inclusion  $\mathcal{H}' \hookrightarrow \mathcal{H}$  preserves the rank.*

(c)  $K_0(\mathcal{H}) = \mathbb{Z}[S] \oplus K_0(\mathcal{H}')$ .

*Proof.* Since  $\mathcal{H}$  is hereditary and  $S$  is exceptional, inclusion  $\mathcal{H}' \hookrightarrow \mathcal{H}$  admits a left adjoint  $\ell$  and a right adjoint  $r$  [4]. Since  $S$  is simple, the category  $\mathcal{H}'_0$  of torsion objects in  $\mathcal{H}'$  agrees with  $\mathcal{H}' \cap \mathcal{H}_0$ , and therefore  $\mathcal{H}'_0 = \coprod_{y \in \mathbb{X}} \mathcal{U}'_y$  with  $\mathcal{U}'_y = \mathcal{U}_y$  if  $y \neq x$ ; further  $\mathcal{U}'_x = S^\perp \cap \mathcal{U}_x$  is connected as well with  $u(x) - 1$  isomorphic classes of simple objects. Moreover,  $w'_y = w_y$  for each  $y \in \mathbb{X}$ , so inclusion  $\mathcal{H}' \hookrightarrow \mathcal{H}$  preserves the rank.

If  $E$  is torsionfree in  $\mathcal{H}$  without self-extensions, then  $\text{Hom}(S, E) = 0$  and  $\ell E$  is given by the middle term of the  $S$ -universal extension

$$0 \rightarrow E \rightarrow \ell E \rightarrow \text{Ext}^1(S, E) \otimes_k S \rightarrow 0,$$

in particular  $\ell E$  is non-zero torsionfree. From the exactness of

$$0 = \text{Ext}^1(E, E) \rightarrow \text{Ext}^1(E, \ell E) \rightarrow \text{Ext}^1(S, E) \otimes_k \text{Ext}^1(E, S) = 0$$

we get  $\text{Ext}^1(E, \ell E) = 0$ . By definition  $\text{Ext}^1(S, \ell E) = 0$  holds, hence  $\text{Ext}^1(\ell E, \ell E) = 0$ , which proves the existence of a torsionfree object in  $\mathcal{H}'$  without self-extensions.

Finally, assume that  $A$  and  $B$  are in  $\mathcal{H}'$ . Since

$$\text{D Ext}^1(A, B) = \text{Hom}(\ell \tau_{\mathcal{H}}^- B, A) = \text{Hom}(B, r \tau_{\mathcal{H}} A),$$

projectivity of  $A$  (resp. injectivity of  $B$ ) in  $\mathcal{H}'$  implies  $r \tau_{\mathcal{H}} A = 0$  (resp.  $\ell \tau_{\mathcal{H}}^- B = 0$ ) which, by the construction of  $r$  and  $\ell$  [4], implies  $A \in \mathcal{H}'_0$  (resp.  $B \in \mathcal{H}'_0$ ). But any projective (resp. injective) in  $\mathcal{U}'_y$ ,  $y \in \mathbb{X}$ , is zero, hence  $A = 0$  (resp.  $B = 0$ ). Invoking Auslander-Reiten theory this implies that  $\tau_{\mathcal{H}'} = r \tau_{\mathcal{H}}$  and  $\tau_{\mathcal{H}'}^- = \ell \tau_{\mathcal{H}}^-$  are inverse equivalences. For assertion (c) we refer to [4]. Further  $\langle [S], K_0(\mathcal{H}') \rangle = 0$  implies that the Euler forms on  $K_0(\mathcal{H})$  and  $K_0(\mathcal{H}')$  do have the same determinant.  $\square$

Repeating the process of Proposition 1  $\sum_{x \in \mathbb{X}} (u(x) - 1)$  times leads to an exact abelian subcategory  $\mathcal{H}'$  of  $\mathcal{H}$  satisfying (iii) (a), (b), (c) with  $u'(x) = 1$  for each  $x \in \mathbb{X}$ .

**Proposition 2.** *Each torsionfree rank one object  $L$  of  $\mathcal{H}'$  is exceptional. For each simple object  $S_x$ ,  $x \in \mathbb{X}$ , we have  $\dim_k \text{Hom}(L, S_x) = 1$  and  $[S_x] = w$ .*

*Moreover there exist torsionfree rank one objects  $L_1, L_2$  such that*

$$(1) \quad \text{Hom}(L_2, L_1) = 0, \quad \dim_k \text{Hom}(L_1, L_2) = 2, \quad \text{Ext}^1(L_i, L_j) = 0 \quad \text{for } i, j = 1, 2.$$

*Proof.* By Lemma 6 the group  $K_0(\mathcal{H}')$  is free abelian of rank two. Fix a torsionfree object  $L$  of rank one, note that  $\text{End}(L) = k$  hence  $\langle [L], [L] \rangle = 1 - g$ , where  $g = \dim_k \text{Ext}^1(L, L)$ , and define the *degree function*  $\text{deg}: K_0(\mathcal{H}') \rightarrow \mathbb{Z}$  by

$$\text{deg } z = \langle [L], z \rangle - (1 - g) \text{rk } z.$$

We thus obtain a Riemann-Roch formula

$$\langle \mathbf{a}, \mathbf{b} \rangle = (1 - g) \text{rk } \mathbf{a} \text{rk } \mathbf{b} + \begin{vmatrix} \text{rk } \mathbf{a} & \text{rk } \mathbf{b} \\ \text{deg } \mathbf{a} & \text{deg } \mathbf{b} \end{vmatrix} \quad \text{for all } \mathbf{a}, \mathbf{b} \in K_0(\mathcal{H}').$$

The existence of an exceptional object  $E$  of  $\mathcal{H}'$  implies  $0 < \langle [E], [E] \rangle = (1 - g)(\text{rk } E)^2$  hence  $g = 0$ , and so  $L$  is exceptional.

Next we choose  $x \in \mathbb{X}$  such that  $d = \text{deg } w'_x$  is minimal. Note that  $d > 0$ . For  $y \in \mathbb{X}$  write  $\text{deg } w'_y = d \cdot q + r$  with  $q, r \in \mathbb{N}$  and  $0 \leq r < d$ . Extending  $L$  by  $S_y$  and  $q$  times by  $S_x$  we obtain torsionfree rank one objects  $L', L''$  with

$$[L'] = [L] + w'_y, \quad [L''] = [L] + q \cdot w'_x,$$

hence

$$\langle [L''], [L'] \rangle = \langle [L], [L] \rangle + \langle [L], w'_y \rangle - q \cdot \langle [L], w_x \rangle \geq 1.$$

We thus arrive at an exact sequence  $0 \rightarrow L'' \rightarrow L' \rightarrow U \rightarrow 0$ , where  $U \in \mathcal{H}'_0$  has degree  $< d$ . This shows  $U = 0$ , hence  $w'_y = q \cdot w'_x$ . In view of Lemma 6 the classes  $[L], [S_x]$  thus form a basis of  $K_0(\mathcal{H}')$ , accordingly the determinant of the Euler form is  $\pm 1 = \langle [L], [S_x] \rangle^2$ , proving  $\dim \text{Hom}(L, S_x) = 1$ , therefore  $\text{deg } S_x = 1$ .

Put  $L_1 = L$ , and let  $L_2$  be the middle term of a non-split exact sequence  $\mu: 0 \rightarrow L_1 \xrightarrow{\xi} L_2 \rightarrow S_x \rightarrow 0$ . Application of  $\text{Hom}(L_1, -)$  to  $\mu$  shows  $\text{Hom}(L_1, L_2) \cong k^2$  and  $\text{Ext}^1(L_1, L_2) = 0$ , while application of  $\text{Hom}(-, L_1)$  leads to  $\text{Hom}(L_2, L_1) = 0$  and  $\text{Ext}^1(L_2, L_1) = 0$ .

Extend  $\xi$  to a  $k$ -basis  $\xi, \eta$  of  $\text{Hom}(L_1, L_2)$ . Invoking the properties of rank and degree, for each  $a \in k$  the cokernel  $T_a$  of  $\eta - a\xi: L_1 \rightarrow L_2$  is seen to be simple. In fact, each simple object  $S_y$  ( $y \neq x$ ) is isomorphic to some  $T_a$ : Setting  $\xi^* = \text{Hom}(\xi, S_y)$ ,  $\eta^* = \text{Hom}(\eta, S_y)$  we see that  $\xi^*$  is an isomorphism because  $x \neq y$ ; moreover, since  $k$  is algebraically closed there exists an eigenvalue  $a \in k$  such that the kernel  $\text{Hom}(T_a, S_y)$  of  $\eta^* - a\xi^*: \text{Hom}(L_2, S_y) \rightarrow \text{Hom}(L_1, S_y)$  is non-zero. Hence  $S_y \cong T_a$ , in particular  $[S_y] = [L_1] - [L_2] = [S_x]$ .  $\square$

**Proposition 3.**  $\mathcal{H}$  is a tilting object  $\Delta$

$$\begin{array}{ccccc}
 & S_1^{[p_1-1]} & \text{---} \dots & \text{---} & S_1^{[1]} \\
 & \eta_1 & & & \\
 \\
 L_1 \begin{array}{c} \xrightarrow{\xi_1} \\ \xleftarrow{\xi_2} \end{array} L_2 & \xrightarrow{\eta_2} & S_2^{[p_2-1]} & \text{---} \dots & \text{---} S_2^{[1]} \\
 & & \vdots & & \vdots \\
 & \eta_t & S_t^{[p_t-1]} & \text{---} \dots & \text{---} S_t^{[1]}
 \end{array}$$

where  $L_1, L_2$  are torsionfree rank one objects satisfying (1), and  $S_i^{[l]}$  is an exceptional torsion object of length  $l$ .

$\Delta$  is a squid algebra in the sense of [10], i.e. equivalent to the path category of the above quiver subject to the relations

$$(2) \quad \eta_1 \circ \xi_1 = 0, \quad \eta_2 \circ \xi_2 = 0, \quad \eta_i \circ (\xi_2 - \lambda_i \xi_1) = 0, \quad i = 3, \dots, t,$$

where  $\lambda_3, \dots, \lambda_t$  are pairwise distinct non-zero elements from  $k$ .

$\Delta$  also has a realization as a tilting object on the weighted projective line  $\mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$  given by the data  $\mathbf{p} = (p_1, \dots, p_t)$ ,  $\boldsymbol{\lambda} = (\lambda_3, \dots, \lambda_t)$ .

*Proof.* Let  $x_1, \dots, x_t$  denote the sequence of pairwise distinct exceptional elements from  $\mathbb{X}$ . We put  $p_i = u(x_i)$ ,  $\mathcal{U}_i = \mathcal{U}_{x_i}$ . Let  $S_i$  be simple in  $\mathcal{U}_i$  and  $\mathcal{H}'$  denote the right perpendicular category with respect to the system

$$(3) \quad \tau^l S_i, \quad l = 0, \dots, p_i - 2, \quad i = 1, \dots, t,$$

and take a system  $L_1, L_2$  of torsionfree rank one objects from  $\mathcal{H}'$  as established in Proposition 2.

By induction on the rank it is easily verified (compare proof of Lemma 6) that each object from  $\mathcal{H}$  is contained in the smallest subcategory of  $\mathcal{H}$  which is closed under extensions, kernels of epimorphisms, cokernels of monomorphisms and contains

$$(4) \quad L_1, L_2, S_i, \tau S_i, \dots, \tau^{p_i-2} S_i, \quad i = 1, \dots, t;$$

in particular (4) generates  $D^b(\mathcal{H})$ .

Let  $S_i^{[n]}$  denote the indecomposable object of  $\mathcal{U}_i$  with top  $S_i$  and length  $n$ . Then the system

$$(5) \quad L_1, L_2, S_i = S_i^{[1]}, S_i^{[2]}, \dots, S_i^{[p_i-1]}, \quad i = 1, \dots, t,$$

generates  $D^b(\mathcal{H})$  as well, moreover satisfies

$$(6) \quad \text{Ext}^1(S_i^{[a]}, S_{i'}^{[a']}) = 0, \quad \text{Ext}^1(S_i^{[a]}, L_j) = 0, \quad \text{Ext}^1(L_j, S_i^{[a]}) = 0$$

for all  $i, i' = 1, \dots, t$ ,  $a = 1, \dots, p_i - 1$ ,  $a' = 1, \dots, p_{i'} - 1$ ,  $j = 1, 2$ .

Invoking  $w_x = w$  for each  $x \in \mathbb{X}$ , we obtain exact sequences

$$0 \rightarrow L_1 \xrightarrow{\xi_i} L_2 \xrightarrow{\bar{\eta}_i} S_i^{[p_i]} \rightarrow 0 \quad (1 \leq i \leq t),$$

defining  $t$  mutually disjoint 1-dimensional subspaces  $k\xi_i$  of  $\text{Hom}(L_1, L_2)$ , thus  $\xi_i = \xi_2 - \lambda_i \xi_1$  for  $2 \leq i \leq t$ , where  $0 = \lambda_2, \lambda_3, \dots, \lambda_t$  are pairwise distinct. Setting  $\eta_i = [L_2 \xrightarrow{\bar{\eta}_i} S_i^{[p_i]} \rightarrow S_i^{[p_i-1]}]$  and calculating dimensions of Hom-spaces by means of the Euler form further establishes that (2) is a complete set of relations for the subcategory  $\Delta$ .

It is known (cf. [8]) and not difficult to prove that  $\Delta$  can be realized as a tilting sheaf on the weighted projective line  $\mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$  arising from the above data  $\mathbf{p} = (p_1, \dots, p_t)$  and  $\boldsymbol{\lambda} = (\lambda_3, \dots, \lambda_t)$  [3]. Since moreover  $\Delta$  has finite global dimension (actually  $\text{gl. dim } \Delta \leq 2$ ) we may invoke Beilinson's lemma [1] to conclude that  $\Delta$  is also a tilting object in  $\mathcal{H}$ .  $\square$

*Proof of (iii)  $\Rightarrow$  (i).* Abbreviating  $\mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$  to  $\mathbb{X}$  from now on, the right derived functor of  $\text{Hom}_{\mathbb{X}}(\Delta, -): \text{coh}(\mathbb{X}) \rightarrow \text{mod}(\Delta)$  induces an equivalence of triangulated categories  $\alpha: D^b(\text{coh}(\mathbb{X})) \rightarrow D^b(\text{mod}(\Delta))$ , similarly the right derived functor of  $\text{Hom}_{\mathcal{H}}(\Delta, -)$  induces an equivalence  $\beta: D^b(\mathcal{H}) \rightarrow D^b(\text{mod}(\Delta))$ . Since  $\alpha$  and  $\beta$  preserve  $\Delta$  the composition  $\phi = \alpha^{-1} \circ \beta: D^b(\mathcal{H}) \rightarrow D^b(\text{coh}(\mathbb{X}))$  preserves the rank and maps  $\mathcal{H}_0$  onto the category  $\text{coh}_0(\mathbb{X})$  of finite length sheaves.

An indecomposable object  $X$  from  $D^b(\mathcal{H})$  lies in  $\mathcal{H}$  if and only if  $X$  is either in  $\mathcal{H}_0$  or has rank  $> 0$  and satisfies  $\text{Hom}(X, \mathcal{H}_0) \neq 0$ . Similarly an indecomposable object  $Y$  from  $D^b(\text{coh}(\mathbb{X}))$  belongs to  $\text{coh}(\mathbb{X})$  if and only if either  $Y$  is in  $\text{coh}_0(\mathbb{X})$  or  $Y$  has rank  $> 0$  and satisfies  $\text{Hom}(Y, \text{coh}_0(\mathbb{X})) \neq 0$ . Because  $\phi(\mathcal{H}_0) = \text{coh}_0(\mathbb{X})$ , and  $\phi$  preserves the rank, it hence follows that  $\phi$  induces an equivalence  $\mathcal{H} \rightarrow \text{coh}(\mathbb{X})$ , thus concluding the proof of the theorem.  $\square$

Given  $\mathcal{H}$  satisfying (ii) or (iii), the theorem implies the existence of a “natural” bijection between the set  $\mathbb{X}$ , parametrizing the decomposition of  $\mathcal{H}_0$  into connected components, and the projective line  $\mathbb{P}_1(k)$ , the point set underlying  $\mathbb{X}$ . To determine the weighted projective line corresponding to  $\mathcal{H}$ , one fixes a simple object  $S$  with  $\tau S \cong S$  further a rank one object  $L$  of  $\mathcal{H}$ , and forms a non-split extension  $0 \rightarrow L \rightarrow \bar{L} \rightarrow S \rightarrow 0$ . The  $k$ -space  $\text{Hom}(L, \bar{L})$  has dimension two; each non-zero  $v: L \rightarrow \bar{L}$  is a monomorphism whose cokernel is indecomposable of finite length, therefore belongs to  $\mathcal{U}_{\psi(v)}$  for a unique  $\psi(v) \in \mathbb{X}$ . The induced mapping  $\mathbb{P}_1(k) = \mathbb{P}(\text{Hom}(L, \bar{L})) \rightarrow \mathbb{X}$ ,  $kv \mapsto \psi(v)$ , is a bijection turning the function  $u$  from Lemma 1 into a weight function on  $\mathbb{P}_1(k)$ , determining thus completely the data for the weighted projective line.

#### REFERENCES

1. A. A. Beilinson, *Coherent sheaves on  $P_n$  and problems of linear algebra*, *Funct. Anal. Appl.* **12** (1979), 214–216. MR **80c**:14010b
2. P. Gabriel, *Indecomposable representations II*, *Symposia Mat. Inst. Naz. Alta Mat.* **11** (1973), 81–104. MR **49**:5132
3. W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*. In *Singularities, representations of algebras, and vector bundles*. *Lect. Notes Math.* 1273 (1987), 265–297, Springer-Verlag, Berlin-Heidelberg-New York. MR **89b**:14049
4. ———, *Perpendicular categories with applications to representations and sheaves*, *J. Algebra* **144** (1991), 273–343. MR **93b**:16011

5. D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Math. Soc. Lect. Notes **119**, Cambridge University Press, 1988. MR **89e**:16035
6. D. Happel, I. Reiten and S. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. **120** (1996), no. 575. CMP 96:08
7. D. Happel and C. M. Ringel, *Tilted Algebras*, Trans. Amer. Math. Soc. **274** (1982), 399–443. MR **84d**:16027
8. H. Lenzing and H. Meltzer, *Tilting sheaves and concealed-canonical algebras*, CMS Conf. Proc. **18** (1996), 455–473. CMP 96:12
9. H. Lenzing and J. A. de la Peña, *Algebras with a separating tubular family*, Math. Z., to appear.
10. C. M. Ringel, *The canonical algebras* with an appendix by William Crawley-Boevey. In *Topics in Algebra*, Banach Center Publ. **26** (1990), 407–432. MR **93e**:16022

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